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ELECTROMAGNETIC INDUCTION IN A SEMI-INFINITE CONDUCTOR WITH A PLANE BOUNDARY

By A. T. PRICE (*Imperial College, London*)

[Received 14 March 1950]

SUMMARY

The general theory of the induction and of the free decay of electric currents in a semi-infinite conductor with a plane boundary is considered. The solutions of the fundamental equations are of two types, which correspond to two distinct modes of decay of currents in the conductor. The properties of these modes are investigated. Detailed calculations are made of the induced field and currents produced by a periodic or aperiodic line current parallel to the face of the conductor, and the method of calculation for any other inducing fields is indicated. An explanation is given of the indeterminate character of certain problems.

1. Introduction

THE theory of the electric currents and magnetization induced in a permeable conducting sphere by a varying magnetic field has been the subject of numerous investigations by Lamb (**1**), Price (**2**), Chapman and Price (**3**), Lahiri and Price (**4**), Elsasser (**5**), and in many earlier references quoted in (4). The main object of these investigations has been to obtain results applicable to various geophysical problems (**6**). These results are expressed in terms of spherical harmonics, so that a necessary condition for the practical application of the theory is that the field considered shall be expressible by a series of spherical harmonics which is fairly rapidly convergent. This rules out the possibility of considering in this way fields of decidedly local character, since these would involve many harmonics of high order. It is obviously possible, however, to obtain some information in such cases by treating the earth as a semi-infinite conductor bounded by a plane.

A number of investigations relating to such a semi-infinite conductor have been made by Niven (**7**), Sommerfeld (**8**), Slichter (**9**), Buchholz (**10**), but these have usually dealt only with some specific problem, or have been specially concerned with the radiation field; a general theory, similar to that developed for the spherical conductor and dealing specially with the induction field (as distinct from the radiation field treated by Sommerfeld and others), does not seem to have been fully considered. This theory is developed in the present paper and the physical interpretation of the two distinct types of fundamental solutions which are found of the relevant equations is examined. The methods of calculating the induced field and

currents corresponding to any inducing magnetic field of external origin are also indicated.

The particular case when the inducing field is produced by a periodic current along a line parallel to the face of the conductor is worked out in detail, and numerical results are obtained which permit the induced field to be found easily for any line current situated at any distance from a conductor of uniform conductivity. The cases of a circular current circuit and of an oscillating magnetic dipole are dealt with by A. N. Gordon in another paper (**11**).

In connexion with geophysical applications of the theory, it is of course obvious that it is not always possible to infer the induced field over a small region on the surface of a conductor (e.g. a restricted region of the earth's surface) from a knowledge of the inducing field over only the same limited region. This is further emphasized in the present theory by the fact that certain induction problems relating to the semi-infinite conductor are indeterminate. These are considered in section 12 and the corresponding restriction on the applicability of the theory to geophysical problems is indicated.

When the inducing field is not periodic, any change in its intensity will induce a distribution of currents in the conductor which subsequently diffuse through the conductor and decay. In the case of a sudden change of the field, the initial distribution of currents will be confined to a thin layer near the surface, but will gradually penetrate to greater depths. This case is worked out in sections 13–16; the result for any other time variation of the field can be obtained from this particular case by a well-known formula.

Since it is necessary to make frequent references to the fundamental field equations and boundary conditions, these are briefly summarized in sections 3 and 4.

2. Notation

The following notation is used throughout:

H, magnetic field intensity	Ω, magnetic scalar potential
B, magnetic induction	ϕ, electric potential
i, conduction current density	ρ, volume density of charge
E, electric field intensity	σ, surface density of charge
D, electric displacement	μ, magnetic permeability
I, intensity of magnetization	κ, electric conductivity
A, magnetic vector potential	ϵ, dielectric constant

All quantities are measured in Gaussian units.

3. General equations of the field

Inside a continuous medium the field vectors satisfy Maxwell's equations

$$\operatorname{curl} \mathbf{H} = \frac{1}{c}(4\pi\mathbf{i}+\dot{\mathbf{D}}), \tag{3.1}$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \tag{3.2}$$

together with

$$\operatorname{div} \mathbf{B} = 0, \qquad \operatorname{div} \mathbf{i} = -\dot{\rho}, \qquad \operatorname{div} \mathbf{D} = 4\pi\rho. \tag{3.3}$$

If the medium is isotropic and uniform, we have also

$$\mathbf{B} = \mu\mathbf{H}, \qquad \mathbf{i} = \kappa\mathbf{E}, \qquad \mathbf{D} = \epsilon\mathbf{E}, \tag{3.4}$$

where μ, κ, and ϵ are constants. In this case it follows from (3.1) that

$$\operatorname{div}(4\pi\kappa/\epsilon)\mathbf{D}+\operatorname{div}\dot{\mathbf{D}} = 0,$$

whence

$$\rho = \rho_0 e^{-(4\pi\kappa/\epsilon)t}, \tag{3.5}$$

showing that any initial *volume* charge distribution in the conductor disperses at a rate uninfluenced by other field conditions. Hence there is no loss of generality in taking ρ zero, and consequently

$$\operatorname{div} \mathbf{E} = 0. \tag{3.6}$$

It then follows, on taking the curl of equations (3.1) and (3.2), that all the components of every field vector satisfy an equation of the form

$$\nabla^2 F = \frac{\mu}{c^2}\left(4\pi\kappa\frac{\partial}{\partial t}+\epsilon\frac{\partial^2}{\partial t^2}\right)F. \tag{3.7}$$

In a conductor, when the time changes in the field are not too rapid, the term $\epsilon\,\partial F/\partial t$ may be ignored compared with $4\pi\kappa F$. The equation (3.7) then reduces to the induction (or diffusion) equation

$$\nabla^2 F = \frac{4\pi\kappa\mu}{c^2}\frac{\partial}{\partial t}F. \tag{3.8}$$

In a dielectric κ is zero or negligibly small, so that (3.7) reduces to the wave equation

$$\nabla^2 F = \frac{\mu\epsilon}{c^2}\frac{\partial^2 F}{\partial t^2}, \tag{3.9}$$

showing that electromagnetic effects are propagated with speed $c/\sqrt{(\epsilon\mu)}$ through the medium. When we are concerned only with the induction field for relatively slow field-variations, the time of travel across any portion of the dielectric, within the region considered, is negligibly small compared with the time taken by the field changes at a point to become effective. Hence we can ignore the term on the right of (3.9), so that in the dielectric we have simply

$$\nabla^2 F = 0. \tag{3.10}$$

The above approximations are, of course, equivalent to ignoring the *magnetic* effects of the displacement current both in the conductor and in the dielectric. It follows from (3.1) that, in the dielectric, curl **H** is zero to the same order of approximation. Hence **H** can be derived (approximately) from a scalar potential, say

$$\mathbf{H} = -\text{grad}\,\Omega, \tag{3.11}$$

where Ω satisfies Laplace's equation since div **H** is zero.

If **H** and **E** are derived from the vector and scalar potentials $(\mathbf{A}, \phi)$ of the field by the usual relations,

$$\mu\mathbf{H} = \text{curl}\,\mathbf{A}, \qquad \mathbf{E} = -\dot{\mathbf{A}}/c - \text{grad}\,\phi \tag{3.12}$$

it is found that both **A** and ϕ satisfy the fundamental equation (3.7) (or the corresponding approximate equations (3.8) or (3.10)) *provided* **A** and ϕ satisfy the relation

$$\text{div}\,\mathbf{A} + \frac{\mu}{c}\left(4\pi\kappa + \epsilon\frac{\partial}{\partial t}\right)\phi = 0. \tag{3.13}$$

The results obtained in this paper were originally expressed in terms of **A** and ϕ, but Dr. A. N. Gordon points out that it is somewhat simpler to obtain them directly in terms of **E** and **H**. This is because **E** and **H** are non-divergent, whereas **A** in general is not.

4. Boundary conditions

At the interface between two media, the tangential components of **E** and of **H**, and the normal component of **B**, are continuous. The corresponding boundary conditions for **A** and ϕ are that the tangential components of **A** and of grad ϕ are continuous. The normal components of **D** and of **i** are not in general continuous, the sum of their components in the outward directions from the interface being $4\pi\sigma$ and $-\dot{\sigma}$, respectively.

The determination of the currents induced in a conductor of any shape by a given external field, or of the free decay of a system of currents in the conductor, involves finding appropriate solutions of equations (3.8) and (3.10) for the components of the field vectors, which must also satisfy the above boundary conditions.

5. Semi-infinite conductor with a plane boundary

Taking rectangular axes, suppose the conductor occupies the half-space $z < 0$, so that its boundary is the plane $z = 0$, and suppose a varying magnetic system, consisting of moving magnetic matter or of electric current systems, is situated in the region $z > h > 0$. In the intermediate region $0 < z < h$ we assume that the medium is non-conducting and non-magnetic.

The elementary solutions of the differential equations and boundary conditions of sections 3 and 4 will be shown to be of two types. In the first type the corresponding current systems in the conductor possess a magnetic field outside it; in the second they do not. When we are concerned solely with the effects produced by a varying external magnetic system, we require solutions of the first type only. For it will be shown that a solution satisfying the differential equations and boundary conditions and containing a part representing the inducing field can be obtained in terms of solutions of the first type. Now if another solution is found also consistent with the inducing field, the difference between the two solutions would also be a solution (since all the equations are linear) and moreover would correspond to a zero inducing field. Hence this difference can only represent a freely decaying system of currents in the conductor. It would consist, in general, of solutions of both types, but would necessarily be zero if the initial current intensity were everywhere zero.

6. The elementary solutions of the equations

Since the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous at the boundary $z = 0$, while the conductivity κ and permeability μ are discontinuous there, it is convenient to express each of these field vectors in terms of elementary solutions of the form $Z(z,t)\mathbf{F}(x,y)$, the variable t occurring only in Z because $\partial/\partial t$ is multiplied by $\kappa\mu$ in the relevant equation (3.8).

Thus, taking
$$\mathbf{E} = Z(z,t)\mathbf{F}(x,y) \tag{6.1}$$
and substituting in (3.6), (3.8), and (3.10), we obtain
$$Z\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)+F_z\frac{\partial Z}{\partial z}=0, \tag{6.2}$$
$$\frac{\partial^2\mathbf{F}}{\partial x^2}+\frac{\partial^2\mathbf{F}}{\partial y^2}=\frac{1}{Z}\left(\frac{4\pi\kappa\mu}{c^2}\frac{\partial Z}{\partial t}-\frac{\partial^2 Z}{\partial z^2}\right)\mathbf{F} \quad \text{for} \quad z<0, \tag{6.3}$$
and
$$\frac{\partial^2\mathbf{F}}{\partial x^2}+\frac{\partial^2\mathbf{F}}{\partial y^2}=-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\mathbf{F} \quad \text{for} \quad z>0. \tag{6.4}$$

Since the coefficient of $\mathbf{F}$ in the term on the right of (6.3) or of (6.4) is independent of x and y, it must be a real constant,* $-\lambda^2$ say, so that
$$\frac{\partial^2\mathbf{F}}{\partial x^2}+\frac{\partial^2\mathbf{F}}{\partial y^2}+\lambda^2\mathbf{F}=0. \tag{6.5}$$

* It will be seen from (6.7) that this constant is necessarily negative.

and
$$\frac{\partial^2 Z}{\partial z^2}-\lambda^2 Z-\frac{4\pi\kappa\mu}{c^2}\frac{\partial Z}{\partial t}=0 \quad \text{for} \quad z<0, \tag{6.6}$$

$$\frac{\partial^2 Z}{\partial z^2}-\lambda^2 Z=0 \quad \text{for} \quad z>0. \tag{6.7}$$

From (6.2) it follows that either

$$F_z=0 \quad \text{and} \quad \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}=0, \tag{6.8}$$

or
$$\frac{1}{F_z}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)=-\frac{1}{Z}\frac{\partial Z}{\partial z}=-\alpha, \tag{6.9}$$

where α is any constant (real or complex).

The two cases corresponding to (6.8) and (6.9), respectively, will now be considered separately.

6.1. Elementary solutions of the first type

In the first case, it is easily seen from (6.8) that $\mathbf{F}$ can be written in the form
$$\mathbf{F}=\left(\frac{\partial P}{\partial y},-\frac{\partial P}{\partial x},0\right), \tag{6.10}$$

where P is a function of x and y, which from (6.5) must satisfy the equation
$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\lambda^2 P=0. \tag{6.11}$$

In the dielectric we have, from (6.7)
$$Z=\mathscr{A}(t)e^{\lambda z}+\mathscr{B}(t)e^{-\lambda z},$$

so that
$$\mathbf{E}=\{\mathscr{A}(t)e^{\lambda z}+\mathscr{B}(t)e^{-\lambda z}\}\left(\frac{\partial P}{\partial y},-\frac{\partial P}{\partial x},0\right) \quad \text{for} \quad z>0, \tag{6.12}$$

and, since the tangential components of $\mathbf{E}$ are continuous at $z=0$, the corresponding expression for $\mathbf{E}$ inside the conductor is
$$\mathbf{E}=Z(z,t)\left(\frac{\partial P}{\partial y},-\frac{\partial P}{\partial x},0\right) \quad \text{for} \quad z<0, \tag{6.13}$$

where
$$Z(-0,t)=\mathscr{A}(t)+\mathscr{B}(t). \tag{6.14}$$

The corresponding magnetic field is readily obtained from equation (3.2), which shows that
$$\mu\dot{\mathbf{H}}=-c\left(\frac{\partial Z}{\partial z}\frac{\partial P}{\partial x},\frac{\partial Z}{\partial z}\frac{\partial P}{\partial y},\lambda^2 ZP\right) \quad \text{for} \quad z<0, \tag{6.15}$$

and
$$\dot{\mathbf{H}}=-c\lambda\,\mathrm{grad}\{(\mathscr{A}(t)e^{\lambda z}-\mathscr{B}(t)e^{-\lambda z})P\} \quad \text{for} \quad z>0. \tag{6.16}$$

These expressions satisfy the boundary condition that the normal component of $\mathbf{B}$ (or $\dot{\mathbf{B}}$) is continuous at $z=0$ in virtue of equation (6.14).

The remaining boundary condition, that the tangential components of **H** are continuous, leads to the further relation

$$Z'(-0,t) = \lambda\mu\{\mathscr{A}(t)-\mathscr{B}(t)\}, \tag{6.17}$$

where Z' denotes $\partial Z/\partial z$.

The above expressions for the field vectors will be called *solutions of the first type*, and it will now be shown that only solutions of this type are required when considering the effects of an external inducing magnetic field. For in this case the field intensity in the conductor must tend to zero as z becomes large and negative, so that

$$Z(z,t) \to 0 \quad \text{as} \quad z \to -\infty. \tag{6.18}$$

Further, the scalar potential Ω of the total magnetic field will satisfy Laplace's equation in the region $0 < z < h$, and can therefore be expressed as a summation of functions of the form

$$\Omega = -\{A(t)e^{\lambda z}+B(t)e^{-\lambda z}\}P(x,y), \tag{6.19}$$

where λ is real and positive (or zero*) and P satisfies equation (6.11).

The function $P(x,y)$ can, in fact, be written in polar coordinates in the form $J_s(\lambda R)\cos(s\theta+\epsilon)$, and an arbitrary function Ω, given over $z = 0$, can be expressed in terms of these functions by means of Fourier's theorem and the Fourier–Bessel integral. For example, the potential at (x,y,z) of a magnetic pole at (x_0,y_0,z_0) can be expressed in the region $0 < z < z_0$ in the form $-m\int_0^\infty e^{\lambda(z-z_0)}J_0(\lambda R)\,d\lambda$, where $R = \sqrt{\{(x-x_0)^2+(y-y_0)^2\}}$.

Now the part of Ω in (6.19) which involves $e^{\lambda z}$ corresponds to sources of the field over the region $z \geqslant h$, i.e. to the given inducing field, while the part involving $e^{-\lambda z}$ corresponds to the field arising in the region $z < 0$, i.e. to the field of the induced currents (and induced magnetism if $\mu \neq 1$) in the conductor. Hence we may regard $A(t)$ in (6.19) as a known function, and on comparing with (6.16) we see that

$$c\lambda\mathscr{A}(t) = -\dot{A}(t), \qquad c\lambda\mathscr{B}(t) = \dot{B}(t), \tag{6.20}$$

so that $\mathscr{A}(t)$ is known.

The equations (6.14), (6.17), and (6.18), together with (6.6) then serve to determine $\mathscr{B}(t)$ and $Z(z,t)$ uniquely in terms of the given $A(t)$. The induced field will therefore be completely determined if the initial condition (i.e. the initial value of $B(t)$) is known. The induced field (and current distribution) is thus expressed entirely in terms of solutions of the first type. In these solutions the z-component of **E**, and therefore also of **i**, is everywhere zero. We have therefore the important result that *the induced currents flow everywhere parallel to the surface of the conductor.*

* The case $\lambda = 0$ requires special treatment and is considered in section 11.

7. Elementary solutions of the second type

The second of the two possibilities mentioned in section 6, namely, the case when equations (6.9) are satisfied, will now be considered. These equations show that Z must be of the form

$$Z = C(t)e^{\alpha z}. \tag{7.1}$$

Hence, inside the conductor we have, from (6.6),

$$(\alpha^2-\lambda^2)C(t) = \frac{4\pi\kappa\mu}{c^2}\dot{C}(t),$$

so that

$$C(t) = C_0 e^{-\beta t}, \tag{7.2}$$

where

$$\beta = \frac{c^2(\lambda^2-\alpha^2)}{4\pi\kappa\mu}. \tag{7.3}$$

From (6.9) and (6.5) it is easily shown that

$$F_x = \frac{\alpha}{\lambda^2}\frac{\partial F_z}{\partial x}+\frac{\partial P_1}{\partial y}, \qquad F_y = \frac{\alpha}{\lambda^2}\frac{\partial F_z}{\partial y}-\frac{\partial P_1}{\partial x},$$

where P_1 is a function of x and y satisfying (6.11). The terms involving P_1 obviously lead again to solutions of the first type and can therefore be omitted. Also from (6.5) the component F_z satisfies (6.11). Hence, writing $F_z = \lambda P$, we have a second type of solution for $\mathbf{E}$ of the form

$$\mathbf{E} = C_0 e^{-\beta t+\alpha z}\left(\frac{\alpha}{\lambda}\frac{\partial P}{\partial x}, \frac{\alpha}{\lambda}\frac{\partial P}{\partial y}, \lambda P\right) \quad \text{for} \quad z < 0. \tag{7.4}$$

Outside the conductor, Z satisfies the equation (6.7) and on comparing this with (7.1) we see that $\alpha = \pm\lambda$ for $z > 0$. It then follows that $\mathbf{E}$ is of the form

$$\mathbf{E} = \operatorname{grad}\{A_1(t)e^{\lambda z}+B_1(t)e^{-\lambda z}\}P(x,y) \quad \text{for} \quad z > 0. \tag{7.5}$$

For the corresponding magnetic field inside the conductor we have

$$\mu\dot{\mathbf{H}} = -c\operatorname{curl}\mathbf{E} = -cC_0 e^{-\beta t+\alpha z}(\lambda-\alpha^2/\lambda)\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right) \quad \text{for} \quad z < 0. \tag{7.6}$$

Outside the conductor, since $\mathbf{E}$ is a gradient vector, we have

$$\dot{\mathbf{H}} = 0 \quad \text{for} \quad z > 0. \tag{7.7}$$

Hence, *the varying current distribution associated with the second type of solution has zero magnetic field outside the conductor*. This explains why this type of solution does not enter into the problem of finding the currents induced by a varying external magnetic field.

With regard to the boundary conditions, we first note that the continuity of the normal component of $\dot{\mathbf{B}}$ is already ensured since this component is everywhere zero. To make the tangential components of $\dot{\mathbf{H}}$

continuous (i.e. zero at $z = 0$) we should require C_0 to be zero if α were real. Hence, to get a non-trivial result we must take α and C_0 complex, and then make the real part of $C_0 e^{-\beta t + \alpha z}(\lambda - \alpha^2/\lambda)$ zero at $z = 0$. On writing $C_0 = A_0 + iB_0$ and $\alpha = \phi + i\omega$, it is easily found that this condition requires $A_0 = 0$ and $\phi = 0$. It then follows that β is real and positive, and, on taking the real parts of the expressions (7.4) and (7.6) for $\mathbf{E}$ and $\mathbf{H}$, we obtain

$$\mathbf{E} = -B_0 e^{-\beta t}\left(\frac{\omega}{\lambda}\cos\omega z\frac{\partial P}{\partial x}, \frac{\omega}{\lambda}\cos\omega z\frac{\partial P}{\partial y}, \lambda P\sin\omega z\right) \quad \text{for} \quad z < 0, \tag{7.8}$$

$$\mu\dot{\mathbf{H}} = cB_0(\lambda + \omega^2/\lambda)e^{-\beta t}\sin\omega z\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right) \quad \text{for} \quad z < 0. \tag{7.9}$$

These clearly satisfy $\operatorname{div}\mathbf{E} = 0$, $\operatorname{div}\mathbf{H} = 0$, which are equivalent to (6.9).

One further boundary condition remains to be satisfied, namely, that the tangential components of $\mathbf{E}$ are continuous. Using (7.5) and (7.8), this gives the relation

$$A_1(t) + B_1(t) = -B_0 e^{-\beta t}\omega/\lambda. \tag{7.10}$$

It will be observed that $A_1(t)$ corresponds to an electric field of external origin, and $B_1(t)$ to the field of a charge distribution on the *surface* of the conductor, since there is no volume charge in virtue of (3.5) and (3.6). Now if we suppose the initial conditions (at $t = 0$, say) are given, all the parameters appearing in the expressions (7.8) and (7.9) will be determined, independently of $A_1(t)$. In particular the time factor $e^{-\beta t}$ will be independent of $A_1(t)$, the equation (7.10) merely serving to determine the surface charge corresponding to $B_1(t)$. It follows that the field and current distribution within the conductor decay exponentially without change of form, *and are unaffected by any external electric field*.

The reason why this result holds is that we have assumed at the outset that the *magnetic* effects of the displacement current $\dot{\mathbf{D}}$ can be ignored. These effects are only important when the field is changing sufficiently rapidly to make $\epsilon\dot{\mathbf{E}}$ comparable with $4\pi\kappa\mathbf{E}$, and to find them it would, of course, be necessary to use the general equation (3.1).

The solutions of the second type which have been obtained above obviously correspond to the decay of current-distributions of special form in the conductor. The decay of current distributions in general will now be considered.

8. The free modes of decay of current in the conductor

The results of the last section show that solutions of the *second* type always correspond to the free decay of certain distributions of electric current in the conductor, these distributions remaining unchanged in form while decreasing exponentially in intensity. There are also freely decaying

current systems corresponding to the *first* type of solution, which similarly remain unchanged in form while decreasing in intensity. These are obtained by putting $A(t)$ (corresponding to external magnetic sources) zero in (6.19). This implies that $\mathscr{A}(t)$ given by (6.20) is also zero; the boundary conditions (6.14), (6.17) then lead to

$$Z'(0,t) = -\lambda\mu Z(0,t). \tag{8.1}$$

Also the solutions of (6.6) which are of the required form $f(z)g(t)$ are easily found to be

$$Z = (a_1 e^{\alpha z} + b_1 e^{-\alpha z}) e^{-\beta t}, \tag{8.2}$$

where β is again given by (7.3), and α is a constant, which, in order to make the time factor $g(t)$ real, must be either purely real or purely imaginary. If α is real we can take it as positive without loss of generality, and equation (8.2) then shows that b_1 must be zero, since Z must not tend to infinity as $z \to -\infty$. But, if b_1 is zero, (8.1) requires that $\alpha = -\lambda\mu$, which is impossible since $\lambda\mu$ is essentially positive. Hence α cannot be real. We therefore take $\alpha = i\omega$, where ω is real; equation (8.2) can then be written in the form

$$Z = (C_1 \cos \omega z + D_1 \sin \omega z) e^{-\beta t}, \tag{8.3}$$

where C_1 and D_1 are real. The boundary condition (8.1) now requires

$$D_1 \omega = -\lambda\mu C_1. \tag{8.4}$$

Hence the current density at any point inside the conductor is given by

$$\mathbf{i} = \kappa\mathbf{E} = \kappa C_1 e^{-\beta t}\left(\cos \omega z - \frac{\lambda\mu}{\omega} \sin \omega z\right)\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right), \tag{8.5}$$

where

$$\beta = c^2(\lambda^2 + \omega^2)/4\pi\kappa\mu. \tag{8.6}$$

Outside the conductor, equations (6.12) and (6.14) show that the electric field is

$$\mathbf{E} = C_1 e^{-\beta t - \lambda z}\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right). \tag{8.7}$$

The corresponding magnetic field is found from (6.15) and (6.16); these equations give on integrating with respect to t,

$$\mu\mathbf{H} = -\frac{cC_1}{\beta} e^{-\beta t}\left\{(\omega \sin \omega z + \lambda\mu \cos \omega z)\left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\right), \frac{\lambda^2}{\omega}(\lambda\mu \sin \omega z - \omega \cos \omega z) P\right\}$$

$$\text{for} \quad z < 0, \tag{8.8}$$

and

$$\mathbf{H} = -\frac{cC_1\lambda}{\beta} e^{-\beta t} \operatorname{grad}(e^{-\lambda z} P) \quad \text{for} \quad z > 0. \tag{8.9}$$

Equations (8.5) to (8.9) give the free modes of decay of the *first type*. In these modes the currents all flow parallel to the bounding plane. There are modes corresponding to all real values of λ and ω from 0 to ∞, and to

all functions $P(x, y)$ which satisfy (6.11), the distribution of current density in each mode being a simple periodic function of z.

The free modes of the *second type* are given by equation (7.8) and the related equations of section 7. To obtain expressions for the current distribution and field in a form directly comparable with those given above for the modes of the first type, we write $C_2 = -B_0\omega/\lambda$ in (7.8), so that the current density at any point in the conductor is given by

$$\mathbf{i} = \kappa\mathbf{E} = \kappa C_2 e^{-\beta t}\left(\cos\omega z\frac{\partial P}{\partial x}, \cos\omega z\frac{\partial P}{\partial y}, \frac{\lambda^2 P}{\omega}\sin\omega z\right), \tag{8.10}$$

and the electric field outside the conductor is given by

$$\mathbf{E} = C_2 e^{-\beta t}\operatorname{grad}(e^{-\lambda z}P). \tag{8.11}$$

The corresponding magnetic field inside the conductor is easily found to be

$$\mathbf{H} = \frac{4\pi\kappa}{\omega}C_2 e^{-\beta t}\sin\omega z\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right), \tag{8.12}$$

while outside the conductor there is no magnetic field.

In these second type modes there is, in general, a component of the current density in the direction normal to the boundary plane $z = 0$, though this component is always zero at the boundary itself. There are, just as in the first type, modes corresponding to all values of λ and ω between 0 and ∞, and to all functions $P(x, y)$ satisfying (6.11). Also the current density is again a simple periodic function of z.

There is, however, one peculiarity about these modes which requires explanation. The electric field outside the conductor, given by (8.11), corresponds to a distribution of charge on the *surface* $z = 0$, since by (3.6) there can be no volume distribution of charge in the conductor. This surface charge distribution is given by

$$\sigma = E_z/4\pi = -\frac{\lambda}{4\pi}C_2 e^{-\beta t}P(x, y), \tag{8.13}$$

and, since it varies with the time, it implies a normal component of current at the surface of the conductor of amount

$$i_z = \dot{\sigma} = \frac{\dot{E}_z}{4\pi} = \frac{\lambda\beta}{4\pi}C_2 e^{-\beta t}P(x, y), \tag{8.14}$$

in apparent contradiction with (8.10), which makes i_z zero at $z = 0$. The value of i_z given by (8.14) is, however, of the same order of magnitude as the displacement current $\dot{\mathbf{D}}/4\pi$, which we have already assumed can be neglected in comparison with $\mathbf{i}$ in the fundamental equation (3.1). Hence there is no real contradiction in the above results. Equation (8.10) is, in fact, an approximation, which holds good for the range of values of β (i.e.

of the conductivity κ, linear dimensions λ^{-1}, ω^{-1}, etc.) contemplated, but which will break down if the value of β is too large.

The physical interpretation is that a minute portion (only) of the (positive or negative) current, flowing towards the boundary in any one of these modes, will carry on right up to the boundary, and there build up a surface charge, whose electric field is sufficient to deflect the remaining, and much greater, part of the current away from the boundary, so that at the boundary itself the current flow is practically parallel to it. The electric field $\mathbf{E}$ due to the surface charge σ will be $4\pi\sigma/\epsilon$, whereas the magnetic field $\mathbf{H}$ of the corresponding current $\dot{\sigma}$ is given by $|\operatorname{curl}\mathbf{H}| = 4\pi\dot{\sigma}/c$. Hence, although the electric field of the surface charge is of fundamental importance in determining the distribution of current in the conductor, the magnetic field of the current associated with this varying charge is negligible.

It is of interest to note that the free modes corresponding to the two different types of solution become identical when $\lambda = 0$. Thus, taking the first type of solution and putting $\lambda = 0$ in (8.5), we obtain

$$\mathbf{i} = \kappa C'_1 e^{-\beta t} \cos \omega z \left(\frac{\partial P_1}{\partial y}, -\frac{\partial P_1}{\partial x}, 0 \right), \tag{8.15}$$

where

$$\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} = 0, \tag{8.16}$$

so that P_1 is the real part of a function, $P_1 + iP_2$ say, of $x+iy$. Hence $\partial P_1/\partial y = -\partial P_2/\partial x$ and $\partial P_1/\partial x = \partial P_2/\partial y$, so that (8.15) can be written

$$\mathbf{i} = -\kappa C'_1 e^{-\beta t} \cos \omega z \left(\frac{\partial P_2}{\partial x}, \frac{\partial P_2}{\partial y}, 0 \right),$$

and this is obviously of the form assumed by the second type of solution (8.10) when $\lambda = 0$. In these particular solutions there is no magnetic field outside the conductor, and the electric field outside is everywhere parallel to the surface and independent of z.

We may also add that the solutions (8.5) give the free modes of decay of current in a uniform plate of thickness h, provided ω is now restricted to the discrete set of values $n\pi/h$, where n is any integer.

9. Periodic inducing field

If the external magnetic system is periodic in the time, it may be analysed into simple harmonic constituents, and the effects of the separate constituents may be superposed in virtue of the linearity of the equations. Taking, therefore, a single harmonic, we now calculate that part of the induced field which has the same harmonic time factor. The remaining part of the induced field will be a transient part depending on how the

external variations were first set up. This part depends on the free modes of decay of current in the conductor; it is negligible if the periodic external system has been in existence a sufficiently long time.

If the period of the inducing system is $2\pi/p$, the equation (6.19) for the potential of the total magnetic field outside the conductor may be written in the form

$$\Omega = -(Ae^{\lambda z} + Be^{-\lambda z})e^{ipt}P(x, y), \tag{9.1}$$

where A and B are complex constants and the real part of the expression on the right is to be taken. The modulus of the ratio B/A will then give the ratio of the amplitude of the induced magnetic field to that of the inducing fields, and the argument of B/A will give the phase difference of the fields.

Since $\mathbf{E}$ must also contain the time-factor e^{ipt} the equation (6.6) reduces to

$$\frac{\partial^2 Z}{\partial z^2} - \left(\frac{4\pi\kappa\mu ip}{c^2} + \lambda^2\right) Z = 0, \tag{9.2}$$

the solution of which is

$$Z = (ae^{\theta z} + be^{-\theta z})e^{ipt}, \tag{9.3}$$

where

$$\theta^2 = 4\pi\kappa\mu ip/c^2 + \lambda^2, \tag{9.4}$$

so that

$$\theta = \frac{1}{\sqrt{2}}\left[\{(\alpha^4+\lambda^4)^{\frac{1}{2}} + \lambda^2\}^{\frac{1}{2}} + i\{(\alpha^4+\lambda^4)^{\frac{1}{2}} - \lambda^2\}^{\frac{1}{2}}\right], \tag{9.5}$$

where

$$\alpha^2 = 4\pi\kappa\mu p/c^2. \tag{9.6}$$

In (9.3) we can obviously take the real part of θ to be positive without loss of generality. The boundary condition (6.18) then shows that b must be zero, and (6.14) and (6.17) together with (6.20) require that

$$c\lambda a = -ip(A - B), \qquad c\theta a = -\mu ip(A + B),$$

so that

$$a = -\frac{2\mu ip}{c(\theta+\lambda\mu)}A, \tag{9.7}$$

$$B = \frac{\theta-\lambda\mu}{\theta+\lambda\mu}A. \tag{9.8}$$

Thus the scalar potential of the induced magnetic field outside the conductor is given by

$$\Omega_i = -\frac{\theta-\lambda\mu}{\theta+\lambda\mu}Ae^{-\lambda z}e^{ipt}P(x, y), \tag{9.9}$$

while the electric field inside the conductor is given by

$$\mathbf{E} = -\frac{2\mu ip}{c(\theta+\lambda\mu)}Ae^{ipt+\theta z}\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right). \tag{9.10}$$

The current density at any point inside the conductor is given by

$$\mathbf{i} = \kappa\mathbf{E} = -\frac{2\kappa\mu ip}{c(\theta+\lambda\mu)} A e^{ipt+\theta z}\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right). \tag{9.11}$$

10. Periodic linear current parallel to surface of conductor

Suppose that the inducing field is produced by a periodic current $cc_0 e^{ipt}$ e.s.u. flowing in the direction of Ox along the line $y = 0$, $z = h$.

The magnetic potential Ω_0 of this current is $2c_0 e^{ipt}\tan^{-1}\{(h-z)/y\}$, and it is easily found by Fourier's integral theorem that this function may be expressed in the form

$$\Omega_0 = -2\int_0^\infty c_0 \exp(ipt+\lambda z-\lambda h)\sin\lambda y\frac{d\lambda}{\lambda}, \quad \text{when} \quad z \leqslant h. \tag{10.1}$$

Comparing this expression with (9.1), we see that $P(x, y)$ is now $\sin\lambda y$ and A is $2c_0 e^{-\lambda h}\, d\lambda/\lambda$ so that, from (9.9), the potential Ω_i of the induced field is

$$\Omega_i = -2\int_0^\infty \frac{\theta-\lambda\mu}{\theta+\lambda\mu} c_0 \exp(ipt-\lambda z-\lambda h)\sin\lambda y\frac{d\lambda}{\lambda}. \tag{10.2}$$

Also, from (9.11), the induced current-density is given by

$$i_x = -2\int_0^\infty \frac{2\kappa\mu ip}{c(\theta+\lambda\mu)} c_0 \exp(ipt-\lambda h+\theta z)\cos\lambda y\, d\lambda, \tag{10.3}$$

with i_y and i_z both zero.

10.1. *The induced field*

The components of the induced field are

$$H_x = 0,$$

$$H_y = -\frac{\partial\Omega_i}{\partial y} = 2c_0\int_0^\infty \frac{\theta-\lambda\mu}{\theta+\lambda\mu}\exp(ipt-\lambda z-\lambda h)\cos\lambda y\, d\lambda,$$

$$H_z = -\frac{\partial\Omega_i}{\partial z} = -2c_0\int_0^\infty \frac{\theta-\lambda\mu}{\theta+\lambda\mu}\exp(ipt-\lambda z-\lambda h)\sin\lambda y\, d\lambda.$$

In these expressions θ is a complex number and the real part of the integrand is to be taken. We can write*

$$H_y+iH_z = 2\int_0^\infty \mathrm{re}\left\{\frac{\theta-\lambda\mu}{\theta+\lambda\mu} c_0 \exp(ipt)\right\}\exp\{-\lambda(h+z+iy)\}\, d\lambda. \tag{10.4}$$

* Note that the direct field of the linear current can similarly be expressed in the form $H_y+iH_z = 2c_0\cos pt/(h-z-iy)$.

Taking c_0 real (so that the inducing current is $c_0 \cos pt$), and assuming the magnetic permeability μ to be unity, we find that

$$\mathrm{re}\left\{\frac{\theta-\lambda\mu}{\theta+\lambda\mu}c_0\exp(ipt)\right\} = c_0\,\mathrm{re}\left\{\left(1-\frac{2i\lambda^2}{\alpha^2}+\frac{2i\lambda\theta}{\alpha^2}\right)\exp(ipt)\right\}$$

$$= c_0\cos pt\left[1-u\sqrt{2}\{(1+u^4)^{\frac{1}{2}}-u^2\}^{\frac{1}{2}}\right]+c_0\sin pt\left[2u^2-u\sqrt{2}\{(1+u^4)^{\frac{1}{2}}+u^2\}^{\frac{1}{2}}\right],$$

where $u = \lambda/\alpha$, and α is given by (9.6).

Hence the expression (10.4) becomes

$$H_y+iH_z = 2c_0\alpha\cos pt\int_0^\infty\left[1-u\sqrt{2}\{(1+u^4)^{\frac{1}{2}}-u^2\}^{\frac{1}{2}}\right]e^{-\zeta u}\,du+$$

$$+2c_0\alpha\sin pt\int_0^\infty\left[2u^2-u\sqrt{2}\{(1+u^4)^{\frac{1}{2}}+u^2\}^{\frac{1}{2}}\right]e^{-\zeta u}\,du, \quad (10.5)$$

where
$$\zeta = \alpha(h+z)+i\alpha y = \xi+i\eta, \text{ say.} \quad (10.6)$$

Asymptotic expansions of the above integrals may be found by expanding the coefficient of the exponential factor in each integrand in ascending powers of u, and integrating term by term. This leads to

$$H_y+iH_z = 2c_0\alpha\cos pt\{\zeta^{-1}-\sqrt{2}(\zeta^{-2}-3\zeta^{-4}+15\zeta^{-6}+315\zeta^{-8}+\ldots)\}+$$

$$+2c_0\alpha\sin pt\{4\zeta^{-3}-\sqrt{2}(\zeta^{-2}+3\zeta^{-4}+15\zeta^{-6}-315\zeta^{-8}+\ldots)\}. \quad (10.7)$$

It will be observed that, when κ (and therefore α) tends to infinity, H_y+iH_z tends to $(2c_0\cos pt)/(h+z+iy)$, so that the induced field is the same as that of a current, equal and opposite to the inducing current, and flowing along the line $y = 0, z = -h$. This is in agreement with Maxwell's image-theory* for a plane conducting sheet.

The expansions obtained above are only of value when $|\zeta|$ is greater than about 6 if results to three significant figures are required. When $|\zeta|$ is less than this it appears necessary to evaluate the integrals numerically. For moderate values of κ, it is precisely these smaller values of ζ for which the calculations need to be made, since the induced field is then only important for sufficiently small values of $h+z$ and y.

If we write
$$H_y = 2c_0\alpha\{Y_1(\xi,\eta)\cos pt+Y_2(\xi,\eta)\sin pt\}, \quad (10.8)$$
$$H_z = 2c_0\alpha\{Z_1(\xi,\eta)\cos pt+Z_2(\xi,\eta)\sin pt\}, \quad (10.9)$$

we find easily from (10.5) that

$$Y_1(\xi,\eta) = \int_0^\infty\left[1-u\sqrt{2}\{(1+u^4)^{\frac{1}{2}}-u^2\}^{\frac{1}{2}}\right]e^{-u\xi}\cos u\eta\,du, \quad (10.10)$$

$$Y_2(\xi,\eta) = \int_0^\infty\left[2u^2-u\sqrt{2}\{(1+u^4)^{\frac{1}{2}}+u^2\}^{\frac{1}{2}}\right]e^{-u\xi}\cos u\eta\,du, \quad (10.11)$$

and $Z_1(\xi,\eta)$, $Z_2(\xi,\eta)$ are similar integrals with $\cos u\eta$ replaced by $\sin u\eta$.

* Clerk Maxwell, *Proc. Roy. Soc.* **20** (1872), 160–8.

It will be observed that a knowledge of the above four functions of the two independent variables (ξ, η) is sufficient to determine the field at any point (y, z) for any set of values of κ, p, and h.

The four functions Y_1, Y_2, Z_1, and Z_2 were calculated for a series of values of (ξ, η) from $(0, 0)$ to $(10, 10)$ and are shown in Table I.

TABLE I

Values of Y_1, Y_2, Z_1, and Z_2

ξ \ η		0	0·5	1·0	2·0	3·0	4·0	5·0	10·0
0	Y_1	+0·471	+0·452	+0·405	+0·291	+0·195	+0·111	+0·065	+0·0146
	Y_2	−0·471	−0·298	−0·169	−0·021	+0·070	+0·082	+0·051	+0·0137
	Z_1	0	−0·092	−0·164	−0·239	−0·271	−0·250	−0·200	−0·1000
	Z_2	0	+0·197	+0·235	+0·189	+0·111	+0·062	+0·032	+0·0040
0·5	Y_1	+0·401	+0·383	+0·348	+0·269	+0·194	+0·125	+0·081	+0·0196
	Y_2	−0·250	−0·213	−0·149	−0·046	+0·026	+0·038	+0·036	+0·0131
	Z_1	0	−0·070	−0·123	−0·189	−0·221	−0·209	−0·185	−0·0983
	Z_2	0	+0·105	+0·145	+0·136	+0·103	+0·068	+0·038	+0·0053
1·0	Y_1	+0·334	+0·327	+0·308	+0·249	+0·195	+0·136	+0·094	+0·0240
	Y_2	−0·176	−0·161	−0·128	−0·052	−0·004	+0·018	+0·028	+0·0123
	Z_1	0	−0·048	−0·092	−0·143	−0·175	−0·175	−0·166	−0·0962
	Z_2	0	+0·053	+0·093	+0·111	+0·093	+0·066	+0·044	+0·0064
2·0	Y_1	+0·250	+0·249	+0·245	+0·211	+0·171	+0·137	+0·104	+0·0320
	Y_2	−0·095	−0·093	−0·089	−0·060	−0·033	−0·001	+0·011	+0·0102
	Z_1	0	−0·029	−0·058	−0·103	−0·126	−0·132	−0·133	−0·0906
	Z_2	0	+0·024	+0·040	+0·057	+0·063	+0·055	+0·040	+0·0081
3·0	Y_1	+0·202	+0·200	+0·196	+0·180	+0·152	+0·129	+0·108	+0·0387
	Y_2	−0·064	−0·061	−0·059	−0·032	−0·010	−0·008	+0·003	+0·0078
	Z_1	0	−0·019	−0·039	−0·075	−0·092	−0·102	−0·108	−0·0843
	Z_2	0	+0·010	+0·020	+0·031	+0·038	+0·041	+0·034	+0·0090
4·0	Y_1	+0·170	+0·168	+0·165	+0·157	+0·140	+0·121	+0·103	+0·0433
	Y_2	−0·048	−0·047	−0·042	−0·029	−0·018	−0·010	−0·004	+0·0059
	Z_1	0	−0·015	−0·028	−0·053	−0·067	−0·080	−0·087	−0·0775
	Z_2	0	+0·006	+0·012	+0·024	+0·027	+0·030	+0·026	+0·0094
5·0	Y_1	+0·148	+0·148	+0·146	+0·141	+0·129	+0·112	+0·098	+0·0467
	Y_2	−0·032	−0·032	−0·030	−0·023	−0·017	−0·012	−0·006	+0·0041
	Z_1	0	−0·011	−0·020	−0·036	−0·050	−0·065	−0·072	−0·0707
	Z_2	0	+0·005	+0·009	+0·017	+0·020	+0·021	+0·020	+0·0093
10·0	Y_1	+0·0863	+0·0862	+0·0857	+0·0839	+0·0811	+0·0774	+0·0731	+0·0499
	Y_2	−0·0106	−0·0105	−0·0103	−0·0097	−0·0087	−0·0075	−0·0062	−0·0009
	Z_1	0	−0·0037	−0·0073	−0·0143	−0·0207	−0·0264	−0·0312	−0·0429
	Z_2	0	+0·0008	+0·0017	+0·0034	+0·0048	+0·0058	+0·0065	+0·0061

For values of $|\xi + i\eta|$ less than 6, the infinite integrals for these functions were evaluated by numerical integration. The actual range of integration taken was from $u = 0$ to $u = 10$, and the ordinates were calculated to five significant figures for intervals of 0·1 from $u = 0$ to $u = 1$, of 0·25 from $u = 1$ to $u = 2$, of 0·5 from $u = 2$ to $u = 5$, and of 1·0 from $u = 5$ to

$u = 10$. The integrals from $u = 0$ to $u = 10$ were then calculated by Simpson's rule and rounded off to three figures. An estimate of the remainder of the integral, viz. the integral from 10 to ∞, was obtained by noting that the expressions in square brackets under the integral signs in (10.10) and (10.11) tend to the values $1/8u^4$ and $-1/4u^2$, respectively, when u becomes large. For example, the remainder of the integral for $Y_1(0, 0)$ is approximately $\int_{10}^{\infty} (1/8u^4)\, du = 0{\cdot}00004$; this is neglected in comparison with the integral from 0 to 10, the value of which is 0·471.

For values of $|\xi+i\eta|$ greater than or equal to 5, the functions were calculated by means of the asymptotic series (10.7). This series gives results correct to three places of decimals (i.e. within about 1 per cent. of the actual value) when $|\xi+i\eta| = 5$, and gives results correct to four places when $|\xi+i\eta| = 10$.

In those cases which were calculated by both methods (i.e. where $|\xi+i\eta|$ is about 5), the results agree to within 1 or 2 per cent. It is believed that the results given in Table I are accurate to the three figures given except when $|\xi+i\eta| \leqslant 2$, in which case the last figure may be doubtful to the extent of 2 or 3.

Values of the functions when ξ or η is greater than 10 can be very quickly obtained from the asymptotic series, which is then rapidly convergent.

The graphs of Y_1, Y_2, Z_1, Z_2, as functions of η, for different values of ξ are shown in Figs. 1 and 2. Since $\xi = (h+z)\sqrt{(4\pi\kappa p)}$ and $\eta = y\sqrt{(4\pi\kappa p)}$, each curve may be regarded as showing the distribution of a component of the induced field over a certain plane parallel to the surface of the conductor, the scale depending on the values of c, h, and $\sqrt{(4\pi\kappa p)}$. For example, if $\sqrt{(4\pi\kappa p)} = 2/h$, the curves $\xi = 2$ give the distribution of the field over the surface $z = 0$, the scale for y being $y = \frac{1}{2}\eta h$, while the curves $\xi = 3$ give the distribution over the plane $z = \frac{1}{2}h$, with the same scale for y.

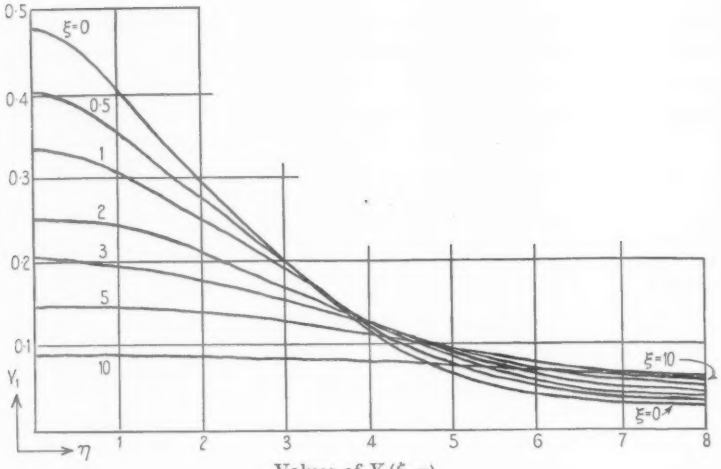
In any given case, the values of h and $\sqrt{(4\pi\kappa p)}$ will fix the ξ which corresponds to the surface $z = 0$ and the scale for y. Each larger value of ξ will then give the distribution of the field over a plane parallel to and above the surface. The curves show how, on moving away from the conductor, the induced field becomes less intense in the immediate neighbourhood of the line current but is spread over a much wider area.

10.2. *The induced current distribution*

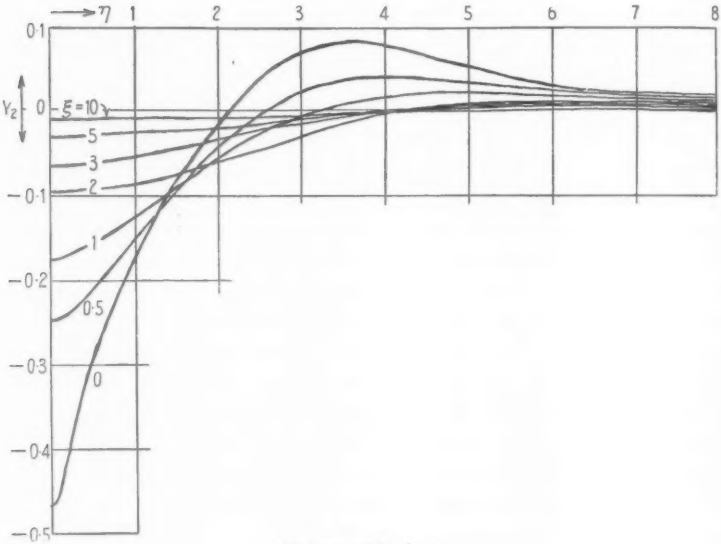
The expression (10.3) for the induced current simplifies to

$$i_x = -\frac{1}{c\pi} \int_0^{\infty} (\theta-\lambda) c_0 \exp(ipt+\theta z-\lambda h) \cos \lambda y \, d\lambda, \qquad (10.12)$$

when $\mu = 1$.



Values of $Y_1(\xi, \eta)$.



Values of $Y_2(\xi, \eta)$.

FIG. 1. Graphs showing the distribution of the tangential component of the induced field over planes parallel to $z = 0$. The component is given by

$$H_y = 4c_0\sqrt{(\pi\kappa p)}\{Y_1(\xi, \eta)\cos pt + Y_2(\xi, \eta)\sin pt\},$$

where $\xi = 2(h+z)\sqrt{(\pi\kappa p)}, \qquad \eta = 2y\sqrt{(\pi\kappa p)}.$

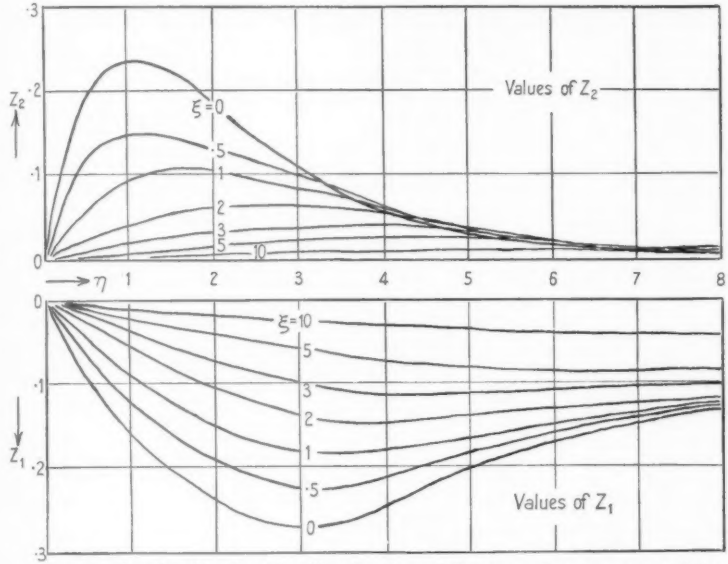


FIG. 2. Graphs showing the distribution of the normal component of the induced field over planes parallel to $z = 0$. The component is given by

$$H_z = 4c_0\sqrt{(\pi\kappa p)}\{Z_1(\xi,\eta)\cos pt + Z_2(\xi,\eta)\sin pt\},$$

where $\xi = 2(h+z)\sqrt{(\pi\kappa p)}, \qquad \eta = 2y\sqrt{(\pi\kappa p)}.$

At the surface ($z = 0$) of the conductor, this becomes

$$i_x = -\frac{c_0\alpha^2}{c\pi\sqrt{2}}\cos pt\int_0^\infty [\{(1+u^4)^{\frac{1}{2}}+u^2\}^{\frac{1}{2}}-u\sqrt{2}]e^{-\alpha hu}\cos\eta u\,du+$$

$$+\frac{c_0\alpha^2}{c\pi\sqrt{2}}\sin pt\int_0^\infty \{(1+u^4)^{\frac{1}{2}}-u^2\}^{\frac{1}{2}}e^{-\alpha hu}\cos\eta u\,du, \quad (10.13)$$

on substituting the value of θ and taking the real part.

The integrals on the right can be computed in terms of the integrals $Y_1(\xi,\eta)$ and $Y_2(\xi,\eta)$ already obtained. For, consider

$$I_1(\xi,\eta) = \int_0^\infty [2u-\sqrt{2}\{(1+u^4)^{\frac{1}{2}}+u^2\}^{\frac{1}{2}}]e^{-\xi u}\cos\eta u\,du.$$

We have

$$\frac{\partial I_1}{\partial\xi} = -\int_0^\infty [2u^2-u\sqrt{2}\{(1+u^4)^{\frac{1}{2}}+u^2\}^{\frac{1}{2}}]e^{-\xi u}\cos\eta u\,du = -Y_2(\xi,\eta).$$

Therefore

$$I_1(\xi,\eta) = \int_\xi^\infty Y_2(\xi,\eta)\,d\xi, \quad (10.14)$$

since $I_1(\xi, \eta) \to 0$ as $\xi \to \infty$. Again consider

$$I_2(\xi, \eta) = \int_0^\infty \sqrt{2}\{(1+u^4)^{\frac{1}{2}} - u^2\}^{\frac{1}{2}} e^{-\xi u} \cos \eta u \, du.$$

Here

$$\frac{\partial I_2}{\partial \xi} = -\int_0^\infty u\sqrt{2}\{(1+u^4)^{\frac{1}{2}} - u^2\}^{\frac{1}{2}} e^{-\xi u} \cos \eta u \, du = Y_1(\xi, \eta) - \frac{\xi}{\xi^2+\eta^2},$$

and therefore

$$I_2(\xi, \eta) = \int_\xi^\infty \left\{\frac{\xi}{\xi^2+\eta^2} - Y_1(\xi, \eta)\right\} d\xi. \tag{10.15}$$

This integral is readily proved convergent by writing $Y_1(\xi, \eta)$ as the real part of the corresponding integral in (10.5). Incidentally we obtain

$$Y_1(\xi, \eta) \to \frac{\xi}{\xi^2+\eta^2} \quad \text{as} \quad |\xi+i\eta| \to \infty. \tag{10.16}$$

The expression (10.13) for the induced current density at the surface may now be written in the form

$$i_x = \frac{c_0 \alpha^2}{2\pi c}\{I_1(\alpha h, \alpha y)\cos pt + I_2(\alpha h, \alpha y)\sin pt\}. \tag{10.17}$$

To determine the current density below the surface it is necessary to compute the integral on the right of (10.12). Apart from a factor α^2, this integral is a function of the three variables αh, αy, and αz.

11. Some special inducing fields; indeterminate problems

As noted in section 6, the fields which correspond to the case $\lambda = 0$ require special consideration. We have already considered, at the end of section 8, the modes of free decay of current systems corresponding to this limiting case. We now consider the nature of the solution when there is a periodic magnetic field of corresponding form arising from external sources. The potential of such a field will be expressible in terms of functions of the form

$$\Omega_0 = -(A_0 z + B_0)e^{ipt}P(x, y), \tag{11.1}$$

where

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0. \tag{11.2}$$

This includes, as a special case, that of a uniform field normal to the surface $z = 0$; its potential is then $-A_0 z e^{ipt}$. Again, if the field is due to a uniform sheet current flowing above and parallel to the surface $z = 0$ in the x-direction, its potential is of the form $-B_0 y e^{ipt}$.

On putting $\lambda = 0$ in (6.6) and assuming Z contains the periodic time factor e^{ipt}, we obtain

$$\frac{\partial^2 Z}{\partial z^2} = \frac{4\pi\kappa\mu ip}{c^2} Z,$$

so that
$$Z = C\exp\{ipt+(1+i)\gamma z\}, \tag{11.3}$$

where
$$\gamma = \sqrt{(2\pi\kappa\mu p/c^2)}, \tag{11.4}$$

the positive value of the square root being taken for γ, because Z must not become infinite as $z \to -\infty$. Hence, from (6.13) the value of $\mathbf{E}$ inside the conductor is given by

$$\mathbf{E} = C\exp\{ipt+(1+i)\gamma z\}\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right). \tag{11.5}$$

Outside the conductor $\mathbf{E}$ must be of the form

$$\mathbf{E} = (az+b)e^{ipt}\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right), \tag{11.6}$$

and the continuity of the tangential components of $\mathbf{E}$ at the boundary requires that
$$C = b. \tag{11.7}$$

Using (6.15) the corresponding magnetic field inside the conductor is found to be

$$\mathbf{H} = -\frac{cC(1-i)\gamma}{p\mu}\exp\{ipt+(1+i)\gamma z\}\left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, 0\right). \tag{11.8}$$

Outside the conductor the total magnetic field must have a potential of the form
$$\Omega = -(Az+B)e^{ipt}P(x,y), \tag{11.9}$$

so that
$$\mathbf{H} = e^{ipt}\left((Az+B)\frac{\partial P}{\partial x}, (Az+B)\frac{\partial P}{\partial y}, AP\right). \tag{11.10}$$

The continuity of the normal component of $\mathbf{B}$ at $z = 0$ then shows that
$$A = 0, \tag{11.11}$$

and the continuity of the tangential components of H requires

$$B = -\frac{c\gamma(1-i)}{p\mu}C. \tag{11.12}$$

Now if we write the potential of the induced field as

$$\Omega_1 = -(A_1 z+B_1)e^{ipt}P(x,y), \tag{11.13}$$

we have $A = A_0+A_1$ and $B = B_0+B_1$. Hence the above equations show that
$$A_1 = -A_0, \tag{11.14}$$

and
$$B_1 = -\frac{c\gamma(1-i)}{p\mu}C-B_0. \tag{11.15}$$

From equation (11.14) we infer that, if the external inducing field given

by (11.1) contains a component normal to the surface of the conductor, this component will always be completely annulled by that of the induced field, which is equal and opposite to it.

On the other hand, equation (11.15) is insufficient to determine the component of the induced field parallel to the surface, because it contains two unknown coefficients B_1 and C. Since no other necessary relation exists between these coefficients, we conclude that, apart from the above result relating to the normal component, the problem of finding the induced field and current system corresponding to an inducing field of the form (11.1) is not a determinate one.

This result is obviously associated with the fact that the conductor extends to infinity, and the field at any point P is determined not only by the currents in the neighbourhood of P, but also by the currents in the outlying portions of the conductor.

12. Conductor treated as a limiting case of a large spherical conductor

Since any actual application of the solution of the present problem must necessarily relate to a restricted portion of the surface of some physically *finite* conductor, it is of interest to treat the conductor as the limit of a finite conductor of particular shape, when its size is increased indefinitely. It is sufficient to consider the case of a spherical conductor of radius a, which will subsequently be made to tend to infinity. In this case the potential of the external inducing field is conveniently expressed in terms of spherical harmonics of the form

$$\Omega_0 = -Ae^{ipt}r^n a^{-n-1}S_n, \tag{12.1}$$

where S_n is a surface harmonic of degree n.

The potential of the corresponding part of the induced field is found to be (see, for example, chap. xxii of reference **6**)

$$\Omega_i = -Be^{ipt}r^{-n-1}a^{n+2}S_n, \tag{12.2}$$

the coefficient B being given by

$$\frac{B}{A} = \frac{n}{n+1}\left[1 - \frac{(2n+1)\mu}{n(\mu-1)+\zeta J_{n-\frac{1}{2}}(\zeta)/J_{n+\frac{1}{2}}(\zeta)}\right], \tag{12.3}$$

where

$$\zeta^2 = -4\pi\kappa\mu ipa^2. \tag{12.4}$$

It follows from (12.3) that

$$\frac{B}{A} \to \frac{n}{n+1} \quad \text{as} \quad a \to \infty. \tag{12.5}$$

Now the normal component of the inducing field at the surface is $nAe^{ipt}S_n$ and that of the induced field is $-(n+1)Be^{ipt}S_n$. Hence, in the

limit as $a \to \infty$ these components cancel each other. This is in agreement with the result of the last section.

On the other hand, the tangential components of the induced and inducing fields are, ultimately, in the ratio $n/(n+1)$. Now *any* surface harmonic of *finite* degree n will lead to a field which is sensibly uniform over a limited region of a sufficiently large spherical surface. If we then confine our attention to this limited region, we might be tempted to treat the conductor as a semi-infinite one, and the field as uniform. But the results of the last section show that, if we do this, the tangential components of the induced field become indeterminate. This is, of course, in agreement with the fact that the above limiting ratio for the tangential components of the induced and inducing fields in the case of the sphere is $n/(n+1)$, and n may be any positive integer, so that this ratio may have any value between $\frac{1}{2}$ and 1. Moreover, by taking conductors of other shapes (e.g. a cylinder), it is easily seen that still other values for this ratio, outside the above range, can also be found.

If in the above we remove the restriction that n remains finite, and allow n and a to tend to infinity together, we can obtain a field which varies sensibly over a region of the spherical surface small enough to be treated as plane. Thus suppose

$$n/a \to \lambda \quad \text{as} \quad a \to \infty. \tag{12.6}$$

Then if S_n is the tesseral harmonic $P_n^s(\cos\theta)\cos(s\phi+\epsilon)$, we find, on considering the limiting values of the expressions (12.1) and (12.2)

$$\Omega_0/a \to -Ae^{ipt}e^{\lambda z}J_s(\lambda R)\cos(s\phi+\epsilon) \tag{12.7}$$

and

$$\Omega_i/a \to -Be^{ipt}e^{-\lambda z}J_s(\lambda R)\cos(s\phi+\epsilon), \tag{12.8}$$

where the z-axis is taken along the line $\theta = 0$, and R is the distance from this axis.

To obtain the limiting value of the ratio B/A given by (12.3), we make use of the recurrence formula for Bessel functions, namely

$$J_{n-\frac{1}{2}}(\zeta)+J_{n+\frac{1}{2}}(\zeta) = \frac{2n+1}{\zeta}J_{n+\frac{1}{2}}(\zeta). \tag{12.9}$$

Writing $\zeta = n\beta$ and denoting the ratio $J_{n+\frac{1}{2}}(\zeta)/J_{n-\frac{1}{2}}(\zeta)$ by ρ_n, this formula gives

$$\frac{1}{\rho_n}+\rho_{n+1} = \frac{2n+1}{n\beta}. \tag{12.10}$$

Hence, if $\rho_n \to \rho$ as $n \to \infty$, we have

$$\frac{1}{\rho}+\rho = \frac{2}{\beta},$$

so that

$$\rho = \beta^{-1}\{1 \pm \sqrt{(1-\beta^2)}\}, \tag{12.11}$$

and the negative sign must be taken since ρ does not become infinite as $\beta \to 0$. Hence from (12.3) we obtain

$$\frac{B}{A} \to 1 - \frac{2\mu}{\mu - 1 + \beta^2\{1-\sqrt{(1-\beta^2)}\}^{-1}} = \frac{\sqrt{(1-\beta^2)}-\mu}{\sqrt{(1-\beta^2)}+\mu} = \frac{\theta-\lambda\mu}{\theta+\lambda\mu},$$

where θ is given by (9.4).

This result is thus identical with the result (9.8), and affords a check on the previous work.

13. Aperiodic inducing fields

When the inducing field has a time factor $A(t)$, which is not simply periodic, the induced field and currents can be determined only if the initial conditions of the field are given. Since, however, the differential equations are linear with respect to t, the effects of different time variations of the inducing field can be superposed, each such variation merely adding a new current distribution to that already existing (though continually decaying) in the conductor. It is therefore sufficient to take, as the initial condition, that the current density is everywhere zero in the conductor. The induced field and current distribution, corresponding to any given $A(t)$, are then determined by finding the particular combination of free modes of the first type (section 6) which, when combined with the particular integral of the equations corresponding to $A(t)$, satisfies this initial condition. This can be done with the aid of the Fourier–Bessel integral, but a more convenient method is to use the operational calculus.

14. The operational method of solution

A formal solution of the differential equations and boundary conditions is first obtained, treating the operator $D \equiv \partial/\partial t$ as an algebraic constant. Thus solving equation (6.6) in this way, we obtain

$$Z(z,t) = e^{\theta z}a(t) + e^{-\theta z}b(t), \tag{14.1}$$

where

$$\theta^2 = 4\pi\kappa\mu D + \lambda^2. \tag{14.2}$$

To satisfy the boundary conditions at $z = 0$ and $z = -\infty$, we find, exactly as in section 9, that $b(t)$ must be zero and

$$a(t) = -\frac{2\mu D}{c(\theta+\lambda\mu)}A(t), \tag{14.3}$$

$$B(t) = \frac{\theta-\lambda\mu}{\theta+\lambda\mu}A(t). \tag{14.4}$$

The coefficients of $A(t)$ in the above two equations may be regarded as operators, which, acting upon $A(t)$, give the required functions $a(t)$ and $B(t)$. The result of any such operation, say

$$a(t) = F(D)A(t), \tag{14.5}$$

is given by
$$a(t) = \frac{d}{dt}\int_0^t A(t-\tau)f(\tau)\,d\tau, \tag{14.6}$$

where $f(t)$ is the solution of the integral equation

$$F(D) = D\int_0^\infty f(t)e^{-Dt}\,dt. \tag{14.7}$$

Thus $a(t)$ and $B(t)$ can be determined from (14.3) and (14.4) without further reference to the differential equations or boundary conditions. The value of $Z(z,t)$ can then be obtained from (14.1) as the result of the operator $e^{\theta z}$ acting on $a(t)$. The induced field and current distribution are thus obtained explicitly as functions of the time.

The function $f(t)$, which is defined by (14.7), is the value of $a(t)$ for the case when the time factor $A(t)$ of the inducing field is Heaviside's unit function $H(t)$, i.e. for the case when the potential of the inducing field increases suddenly by one unit at time $t = 0$. The equation (14.6) is the formal expression of the superposition principle.

15. Evaluation of the induced field

From (14.4) the operator $F(D)$, which is required to obtain the time factor $B(t)$ of the induced field, is $(\theta-\lambda\mu)/(\theta+\lambda\mu)$, where θ is given by (14.2). Taking the case when $\mu = 1$, and expanding $F(D)$ by the binomial theorem in powers of D^{-1}, we have

$$F(D) = 1+\frac{2\lambda^2}{4\pi\kappa D}-\frac{2\lambda D}{\sqrt{(4\pi\kappa D)}}\left\{1+\ldots+\frac{(-)^{n-1}(2n-3)!}{2^n n!\,(n-2)!}\left(\frac{\lambda^2}{4\pi\kappa D}\right)^n+\ldots\right\}. \tag{15.1}$$

Hence, using the known result

$$D^{-m}H(t) = t^m/\Gamma(m+1), \tag{15.2}$$

we find

$$f(t) = 1+\frac{2\lambda^2 t}{4\pi\kappa}-\frac{2\lambda}{\pi}\sqrt{\left(\frac{t}{\kappa}\right)}\left\{1+\ldots+\frac{(-1)^{n-1}}{n!\,(2n-1)(2n+1)}\left(\frac{\lambda^2 t}{4\pi\kappa}\right)^n+\ldots\right\}. \tag{15.3}$$

This series is convergent for all values of t. We can also write the result in the form

$$f(t) = \sqrt{\left(\frac{8}{\pi}\right)}e^{-\frac{1}{2}u^2}D_{-3}(u) \tag{15.4}$$

where $D_n(u)$ is Weber's parabolic cylinder function, and

$$u = \sqrt{\left(\frac{\lambda^2 t}{2\pi\kappa}\right)}. \tag{15.5}$$

Using the asymptotic expansion for $D_n(u)$ we can express $f(t)$ in the alternative form

$$f(t) \sim \sqrt{\left(\frac{8}{\pi}\right)}u^{-3}e^{-\frac{1}{2}u^2}\left\{1-\frac{4!}{2^2u^2}+\ldots+(-)^n\frac{(2n+2)!}{2^{n+1}n!\,u^{2n}}+\ldots\right\}. \tag{15.6}$$

The two expressions (15.3) and (15.6) serve to evaluate $f(t)$ conveniently for all values of t.

16. Aperiodic linear current parallel to the surface

In the case of a line current of strength $C(t)$ along the line $y = 0, z = h$, we can express the potential of its magnetic field in the region $z \leqslant h$, as in section 10, in the form

$$\Omega_0 = -2\int_0^\infty C(t)\exp(\lambda z - \lambda h)\sin\lambda y\, d\lambda/\lambda. \tag{16.1}$$

Hence the potential of the corresponding induced field is

$$\Omega_i = -2\int_0^\infty \frac{\theta-\lambda\mu}{\theta+\lambda\mu} C(t)\exp(-\lambda z - \lambda h)\sin\lambda y\, d\lambda/\lambda, \tag{16.2}$$

and the components of the induced field may be written (cf. section 10) in the form

$$H_y + iH_z = 2\int_0^\infty \frac{\theta-\lambda\mu}{\theta+\lambda\mu} C(t)\exp\{-\lambda(h+z+iy)\}\, d\lambda. \tag{16.3}$$

To determine the effect of a sudden increase of amount C_0 in the line current, we write

$$C(t) = C_0 H(t), \tag{16.4}$$

and we then have from (15.4), taking $\mu = 1$,

$$H_y + iH_z = 8C_0\sqrt{\left(\frac{\kappa}{t}\right)}\int_0^\infty D_{-3}(u)\exp\left\{\sqrt{\left(\frac{2\pi\kappa}{t}\right)}(h+z+iy)u - \tfrac{1}{4}u^2\right\} du. \tag{16.5}$$

If the power series (15.3) is inserted for $\exp(-u^2/4)D_{-3}(u)$ in this integral, we obtain an asymptotic expansion for $H_y + iH_z$, valid for large values of $(h+z+iy)t^{-\frac{1}{2}}$. This gives a convenient method of calculating the field for small values of t. For large values of t, it appears necessary to use the asymptotic expansion (15.6) to evaluate the integrand, and then to compute the integral numerically.

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ON SOME DUAL INTEGRAL EQUATIONS OCCURRING IN POTENTIAL PROBLEMS WITH AXIAL SYMMETRY

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SUMMARY

The use of a Hankel transform can reduce the solution of Laplace's equation in cylindrical coordinates (ρ, z) in the region $0 < \rho < \infty, 0 < z < h$ when the boundary condition on $z = 0$ is a 'mixed' one and that on $z = h$ is of the usual type to the solution of the dual integral equations

$$\left.\begin{aligned} &\int_0^\infty G(u)f(u)J_0(\rho u)\,du = g(\rho) \quad (0 < \rho < 1), \\ &\int_0^\infty f(u)J_0(\rho u)\,du = 0 \qquad\qquad (\rho > 1), \end{aligned}\right\}$$

where $G(u)$, $g(\rho)$ are given functions of the variables indicated and $f(u)$ is to be found. A formal solution of these equations is given and, as an example, the solution is applied to find the potential due to a circular disk at constant potential placed with its plane parallel to, and equidistant from, two earthed parallel plates.

1. Introduction

POTENTIAL problems with axial symmetry for an infinite plate involve the solution of Laplace's equation in cylindrical coordinates (ρ, z) in the region $0 < \rho < \infty, 0 < z < h$ (the plate thickness) under given conditions on the plane boundaries $z = 0$, $z = h$. If the boundary conditions are such that either the potential, its normal derivative, or a linear combination of the potential and its derivative are specified over the *whole* of each plane face of the plate, a convenient method of solution is by the use of the Hankel transform. In such cases the use of this transform eliminates the radial variable and the problem reduces to the solution of a simple differential equation in one independent variable z with known conditions on $z = 0$, $z = h$.

If, however, the condition on one boundary is a *mixed* one, for example, if the potential on $z = 0$ is a prescribed function of ρ inside a certain circle and its normal derivative is zero outside this circle, the solution by a Hankel transform reduces to that of a pair of dual integral equations. A well-known example of this type is the field due to a single circular disk at unit potential. Titchmarsh (**1**) uses the Hankel transform to reduce this

problem to the solution of the dual integral equations†

$$\left.\begin{aligned} \int_0^\infty u^{-1} f(u) J_0(\rho u)\, du &= 1 \quad (0 < \rho < 1), \\ \int_0^\infty f(u) J_0(\rho u)\, du &= 0 \qquad (\rho > 1). \end{aligned}\right\} \tag{1}$$

In this example, if V is the potential, the boundary conditions are

$$\left.\begin{aligned} V &= 1, \quad 0 < \rho < 1, \\ \partial V/\partial z &= 0, \quad \rho > 1, \end{aligned}\right\} \quad \text{when } z = 0, \tag{2}$$

and

$$V \to 0 \quad \text{as} \quad z \to \infty. \tag{3}$$

For the similar problem of the field due to a disk with its plane parallel to, and equidistant from, two parallel plates at zero potential, the boundary conditions are (2) and, instead of (3),

$$V = 0 \quad \text{when} \quad z = h, \tag{4}$$

$2h$ being the distance between the plates. This problem is considered in section 3 and is there reduced to the solution of the dual equations

$$\left.\begin{aligned} \int_0^\infty u^{-1} \tanh hu\, f(u) J_0(\rho u)\, du &= 1 \quad (0 < \rho < 1), \\ \int_0^\infty f(u) J_0(\rho u)\, du &= 0 \quad (\rho > 1). \end{aligned}\right\} \tag{5}$$

The most general integral equations of this type for which a solution is available are, as far as I am aware, the pair considered by Busbridge (2)

$$\left.\begin{aligned} \int_0^\infty u^\alpha f(u) J_\nu(\rho u)\, du &= g(\rho) \quad (0 < \rho < 1), \\ \int_0^\infty f(u) J_\nu(\rho u)\, du &= 0 \qquad (\rho > 1), \end{aligned}\right\} \tag{6}$$

in which $g(\rho)$ is a given function of ρ and α, ν are suitably restricted. These equations are not sufficiently general to include the pair (5) which is typical of potential problems in a region of *finite* thickness with one 'mixed' boundary condition.

† The equations actually derived by Titchmarsh are

$$\left.\begin{aligned} \int_0^\infty F(u) J_0(\rho u)\, du &= 1 \quad (0 < \rho < 1), \\ \int_0^\infty u\, F(u) J_0(\rho u)\, du &= 0 \qquad (\rho > 1), \end{aligned}\right\}$$

which can clearly be put into the form (1) by writing $f(u) = u\, F(u)$.

For potential problems with axial symmetry, only the Bessel function of zero order will be involved, and I have attempted in section 2 a solution of the dual equations

$$\left.\begin{aligned} \int_0^\infty G(u)f(u)J_0(\rho u)\,du &= g(\rho) \quad (0<\rho<1), \\ \int_0^\infty f(u)J_0(\rho u)\,du &= 0 \qquad (\rho>1), \end{aligned}\right\} \tag{7, 8}$$

where $G(u)$, $g(\rho)$ are given functions of the variables indicated and $f(u)$ is to be found. The solution given is a formal one only and the difficult question of convergence is not considered. Under certain circumstances this solution is likely to be suitable for computation purposes. As an example, the potential problem involving the solution of Laplace's equation in cylindrical coordinates under the boundary conditions (2) and (4) is worked out in fair detail in section 3. That the method of solution of equations (7) and (8) could be extended to cover the case when $J_0(\rho u)$ is replaced by $J_\nu(\rho u)$ seems fairly clear but is not attempted here.

2. The solution of the dual integral equations (7) and (8)

We start by considering the integral

$$I(m,k,\rho) = \int_0^\infty u^{1-k}J_{2m+k}(u)J_0(\rho u)\,du, \tag{9}$$

where m is a positive integer (or zero) and k is real and positive. Under these conditions the integral converges for both $0<\rho<1$ and $\rho>1$ and its value is given by Watson (**3***a*). When $\rho>1$, there is a term $\Gamma(-m)$ in the denominator of the factor multiplying the hypergeometric function in Watson's formula and hence the integral vanishes for such values of ρ. By taking

$$f(u) = u^{1-k}\sum_{m=0}^{\infty} a_m J_{2m+k}(u), \tag{10}$$

and by the use of the result just stated, we see that the second of the dual equations (8) is satisfied by this choice of $f(u)$.

The coefficients a_m have now to be chosen so that the form of $f(u)$ assumed in (10) satisfies the first equation (7). For this purpose we need a preliminary result and we first consider the value of $I(m,k,\rho)$ for $0<\rho<1$. Again quoting from Watson (**3***a*), we have

$$I(m,k,\rho) = \frac{\Gamma(m+1)}{2^{k-1}\Gamma(m+k)}\,{}_2F_1(m+1, 1-k-m; 1; \rho^2) \quad (0<\rho<1).$$

The hypergeometric function can be expressed in terms of a Jacobi

polynomial by first using the transformation (see, for instance, Magnus and Oberhettinger, **4***a*)

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z). $$

This gives

$$ I(m,k,\rho) = \frac{\Gamma(m+1)}{2^{k-1}\Gamma(m+k)}(1-\rho^2)^{k-1}\,{}_2F_1(-m,m+k;1;\rho^2) \quad (0<\rho<1). $$

Introducing Jacobi's polynomial defined by

$$ \mathscr{F}_m(\alpha,\gamma,x) = {}_2F_1(-m,\alpha+m;\gamma;x) $$

(Magnus and Oberhettinger, **4***b*), this can be written

$$ I(m,k,\rho) = \frac{\Gamma(m+1)}{2^{k-1}\Gamma(m+k)}(1-\rho^2)^{k-1}\mathscr{F}_m(k,1,\rho^2) \quad (0<\rho<1). \tag{11} $$

By applying Hankel's inversion formula to (9), using (11) and the fact that $I(m,k,\rho)=0$ for $\rho>1$, we have, as the preliminary result required,

$$ u^{-k}J_{2m+k}(u) = \frac{\Gamma(m+1)}{2^{k-1}\Gamma(m+k)}\int_0^1 \rho(1-\rho^2)^{k-1}\mathscr{F}_m(k,1,\rho^2)J_0(u\rho)\,d\rho. \tag{12} $$

Substituting from (10) in (7), we see that a_m has to be chosen to satisfy

$$ \sum_{m=0}^{\infty} a_m \int_0^{\infty} G(u)u^{1-k}J_{2m+k}(u)J_0(\rho u)\,du = g(\rho) \quad (0<\rho<1). $$

Multiplying by $\rho(1-\rho^2)^{k-1}\mathscr{F}_n(k,1,\rho^2)$, where n is a positive integer or zero, and integrating with respect to ρ between 0 and 1 and assuming that the order of integration can be interchanged we have

$$ \sum_{m=0}^{\infty} a_m \int_0^{\infty} G(u)u^{1-k}J_{2m+k}(u)\,du \int_0^1 \rho(1-\rho^2)^{k-1}\mathscr{F}_n(k,1,\rho^2)J_0(u\rho)\,d\rho $$

$$ = \int_0^1 g(\rho)\rho(1-\rho^2)^{k-1}\mathscr{F}_n(k,1,\rho^2)\,d\rho. \tag{13} $$

Writing, for brevity,

$$ E(n,k) = \frac{\Gamma(n+1)}{2^{k-1}\Gamma(n+k)}\int_0^1 g(\rho)\rho(1-\rho^2)^{k-1}\mathscr{F}_n(k,1,\rho^2)\,d\rho, \tag{14} $$

and using (12), (13) gives

$$ \sum_{m=0}^{\infty} a_m \int_0^{\infty} G(u)u^{1-2k}J_{2m+k}(u)J_{2n+k}(u)\,du = E(n,k), \tag{15} $$

which is a set of simultaneous equations to determine the coefficients a_m.

Since (Watson, **3***b*)

$$ \int_0^{\infty} u^{-1}J_{2m+k}(u)J_{2n+k}(u)\,du = 0 \quad (n\neq m), $$

$$ = (4n+2k)^{-1} \quad (n=m), \tag{16} $$

equations (15) give simple expressions in closed form for a_m when

$$G(u) = u^{2k-2}.$$

Apart from being real and positive, k is at our disposal and the solution of equations (15) is facilitated by trying to choose k so that the difference $G(u)-u^{2k-2}$ is fairly small. For the type of application we have in mind this is generally possible, for $G(u)$ will usually contain, besides u, a parameter h and will behave like a power of u for a certain value of h. For example, in the pair of equations (5), $G(u) = u^{-1}\tanh hu$ and $G(u)$ behaves like u^{-1} for large h. If then we are interested in the solution of (5) for large h, we take $k = \frac{1}{2}$ and rearrange (15) so that the difference $G(u)-u^{-1}$ is involved. Returning to the general case, we write (15) in the form

$$\sum_{m=0}^{\infty} a_m \left[\int_0^{\infty} u^{-1} J_{2m+k}(u) J_{2n+k}(u)\, du + \right.$$
$$\left. + \int_0^{\infty} \{u^{2-2k}G(u) - 1\} u^{-1} J_{2m+k}(u) J_{2n+k}(u)\, du \right] = E(n,k). \quad (17)$$

Writing

$$L_{m,n} = (4n+2k) \int_0^{\infty} \{u^{2-2k}G(u) - 1\} u^{-1} J_{2m+k}(u) J_{2n+k}(u)\, du, \quad (18)$$

and using (16), we have

$$a_n + \sum_{m=0}^{\infty} L_{m,n} a_m = (4n+2k) E(n,k). \quad (19)$$

By a proper choice of k the values of $L_{m,n}$ can be kept fairly small, and the coefficients a_m can be found from the set of equations (19) by an iterative process. Expressions for the a_m calculated in this way are necessarily somewhat complicated in the general case and it does not seem worth while to give them here. That the method outlined can give a useful solution is illustrated in the example considered in the next section.

3. An example

Here we find a solution for the potential due to a circular disk at unit potential placed with its plane parallel to and equidistant from two earthed infinite parallel plates. The form of the solution is well suited to numerical computation when the parallel plates are fairly far apart.

We use cylindrical coordinates (ρ, z), take the disk as $z = 0$, $\rho < 1$ and the plates as $z = \pm h$. If V is the potential we have to satisfy

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2} = 0, \quad (20)$$

with

$$V = 1, \quad 0 < \rho < 1; \qquad \partial V/\partial z = 0, \quad \rho > 1, \quad \text{when} \quad z = 0 \quad (21)$$

since there is symmetry about the plane $z = 0$, and

$$V = 0, \quad 0 < \rho < \infty, \quad \text{when} \quad z = h. \tag{22}$$

The same solution applies to the steady temperature in an infinite plate of thickness h when its lower surface is maintained at zero temperature while, on its upper surface, the temperature is kept at unity inside a circle of unit radius and there is no loss of heat outside this circle.

Let $\overline{V}$ be the Hankel transform of V, i.e.

$$\overline{V} = \int_0^\infty \rho V J_0(u\rho)\, d\rho, \tag{23}$$

then

$$\frac{\partial^2 \overline{V}}{\partial z^2} = \int_0^\infty \rho \frac{\partial^2 V}{\partial z^2} J_0(u\rho)\, d\rho = -\int_0^\infty \left(\rho \frac{\partial^2 V}{\partial \rho^2} + \frac{\partial V}{\partial \rho}\right) J_0(u\rho)\, d\rho,$$

using (20); also

$$\int_0^\infty \rho \frac{\partial^2 V}{\partial \rho^2} J_0(u\rho)\, d\rho = -\int_0^\infty \frac{\partial V}{\partial \rho} \{J_0(u\rho) + u\rho J_0'(u\rho)\}\, d\rho,$$

integrating by parts; hence

$$\frac{\partial^2 \overline{V}}{\partial z^2} = \int_0^\infty \frac{\partial V}{\partial \rho} u\rho J_0'(u\rho)\, d\rho = -u \int_0^\infty V\{J_0'(u\rho) + u\rho J_0''(u\rho)\}\, d\rho,$$

integrating by parts again. Using Bessel's equation, the expression in $\{\ \}$ in the last integral is $-u\rho J_0(u\rho)$ and we have

$$\partial^2 \overline{V}/\partial z^2 = u^2 \overline{V}.$$

The solution of this equation is

$$\overline{V} = Ae^{uz} + Be^{-uz},$$

and since from (22), $\overline{V} = 0$ when $z = h$, $A = -Be^{-2uh}$. Hence

$$\overline{V} = B\{e^{-uz} - e^{u(z-2h)}\}.$$

The inversion formula for (23) gives

$$V = \int_0^\infty uB\{e^{-uz} - e^{u(z-2h)}\} J_0(\rho u)\, du, \tag{24}$$

and substitution in the boundary condition on $z = 0$, equations (21), give

$$\left.\begin{aligned} \int_0^\infty uB\{1 - e^{-2hu}\} J_0(\rho u)\, du &= 1 \quad (0 < \rho < 1), \\ \int_0^\infty u^2 B\{1 + e^{-2hu}\} J_0(\rho u)\, du &= 0 \quad (\rho > 1). \end{aligned}\right\}$$

Writing
$$f(u) = u^2 B\{1+e^{-2hu}\}, \tag{25}$$
we have for solution the dual integral equations

$$\left.\begin{aligned} &\int_0^\infty u^{-1}\tanh hu f(u) J_0(\rho u)\,du = 1 \quad (0<\rho<1),\\ &\int_0^\infty f(u) J_0(\rho u)\,du = 0 \qquad\qquad (\rho>1). \end{aligned}\right\} \tag{26}$$

These are a special case of the more general equations already discussed in which $G(u) = u^{-1}\tanh hu$ and $g(\rho) = 1$; as $G(u)$ tends to u^{-1} when h tends to infinity, and we are interested in a solution for large h, we take $k = \frac{1}{2}$. From (10), (19), (18), and (14) the solution is given by

$$f(u) = u^{\frac{1}{2}} \sum_{m=0}^{\infty} a_m J_{2m+\frac{1}{2}}(u), \tag{27}$$

where the coefficients a_m are to be calculated from

$$a_n + \sum_{m=0}^{\infty} L_{m,n} a_m = (4n+1) E(n, \tfrac{1}{2}), \tag{28}$$

in which

$$L_{m,n} = (4n+1)\int_0^\infty \{\tanh hu - 1\} u^{-1} J_{2m+\frac{1}{2}}(u) J_{2n+\frac{1}{2}}(u)\,du, \tag{29}$$

and

$$E(n, \tfrac{1}{2}) = \{2^{\frac{1}{2}}\Gamma(n+1)/\Gamma(n+\tfrac{1}{2})\}\int_0^1 \rho(1-\rho^2)^{-\frac{1}{2}} \mathscr{F}_n(\tfrac{1}{2}, 1, \rho^2)\,d\rho. \tag{30}$$

Now (see Magnus and Oberhettinger, **4** *a*), $\mathscr{F}_0(\alpha, \gamma, x) = 1$ and

$$\int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma}\mathscr{F}_m(\alpha, \gamma, x)\mathscr{F}_n(\alpha, \gamma, x)\,dx = 0 \quad (m \neq n),$$

so that $E(n, \frac{1}{2})$ vanishes for $n > 0$, while for $n = 0$

$$E(0, \tfrac{1}{2}) = (2/\pi)^{\frac{1}{2}}\int_0^1 \rho(1-\rho^2)^{-\frac{1}{2}}\,d\rho = (2/\pi)^{\frac{1}{2}}.$$

Thus the set (28) becomes

$$\left.\begin{aligned} &a_0 + \sum_{m=0}^{\infty} L_{m,0} a_m = (2/\pi)^{\frac{1}{2}},\\ &a_n + \sum_{m=0}^{\infty} L_{m,n} a_m = 0 \quad (n>0). \end{aligned}\right\} \tag{31}$$

The iterative solution of this set is

$$a_n = (2/\pi)^{\frac{1}{2}}[\delta_n - C_n + C'_n - C''_n + \ldots], \tag{32}$$

where $\delta_n = 1$ for $n = 0$, $\delta_n = 0$ for $n > 0$ and

$$\left.\begin{aligned} C_n &= L_{0,n}, \\ C'_n &= \sum_{m=0}^{\infty} L_{m,n} C_m, \\ C''_n &= \sum_{m=0}^{\infty} L_{m,n} C'_m, \\ &\cdots\cdots \end{aligned}\right\} \tag{33}$$

and so on.

A convenient way of evaluating $L_{m,n}$ is to obtain it as a series of powers of $1/h$. Writing the hyperbolic function in (29) as a series of exponentials, we have

$$L_{m,n} = (8n+2) \sum_{s=1}^{\infty} (-1)^s \int_0^{\infty} e^{-2shu} J_{2m+\frac{1}{2}}(u) J_{2n+\frac{1}{2}}(u) u^{-1}\, du.$$

The infinite integral can be expressed as a power series in $1/(2sh)$ by using a result given by Watson (**3** *c*). If we do this, interchange the order of summation and use the result (Whittaker and Watson (**5**))

$$\left.\begin{aligned} \sum_{s=1}^{\infty} (-1)^{s-1} s^{-p} &= (1-2^{1-p})\zeta(p) \quad (p > 1), \\ &= \log_e 2 \qquad\qquad (p = 1), \end{aligned}\right\}$$

where $\zeta(p)$ is Riemann's zeta function, we find

$$L_{m,n} = (8n+2) \sum_{r=0}^{\infty} (-1)^{r-1} \times$$
$$\times \frac{\Gamma(2m+2n+2r+1)\Gamma(2m+2n+2r+2)(1-2^{-2m-2n-2r})}{2^{4m+4n+4r+2}(r)!\,\Gamma(2m+r+\frac{3}{2})\Gamma(2n+r+\frac{3}{2})\Gamma(2m+2n+r+2)} \times$$
$$\times \frac{\zeta(2m+2n+2r+1)}{h^{2m+2n+2r+1}}. \tag{34}$$

For $L_{0,0}$ the first term of the series involves the limit as p approaches unity of $(1-2^{1-p})\zeta(p)$; this, of course, must be replaced by $\log_e 2$.

If h is so large that powers of $1/h$ above the third may be neglected we find from (34)

$$-\tfrac{1}{2}\pi L_{0,0} = (\log_e 2)/h - \zeta(3)/(8h^3), \qquad -\tfrac{1}{2}\pi L_{0,1} = \zeta(3)/(8h^3),$$
$$-\tfrac{1}{2}\pi L_{1,0} = \zeta(3)/(40h^3),$$

and that all other L's are negligible. Substitution in (33) then gives

$$C_0 = -(2/\pi)[(\log_e 2)/h - \zeta(3)/(8h^3)],$$
$$C_1 = -(2/\pi)\zeta(3)/(8h^3),$$
$$C'_0 = (4/\pi^2)(\log_e 2)^2/h^2,$$
$$C''_0 = -(8/\pi^3)(\log_e 2)^3/h^3,$$

and all other C's negligible to this order. Equations (32) then give

$$a_0 = (2/\pi)^{\frac{1}{2}}[1+(2\log_e 2)/(\pi h)+ \\ +(2\log_e 2)^2/(\pi h)^2+(2\log_e 2)^3/(\pi h)^3-2\zeta(3)/(8\pi h^3)],$$

$$a_1 = (2/\pi)^{\frac{1}{2}}\zeta(3)/(8h^3), \tag{33}$$

$$a_2 = a_3 = \ldots = 0.$$

To this order, (27) gives

$$f(u) = u^{\frac{1}{2}}\{a_0 J_{\frac{1}{2}}(u)+a_1 J_{\frac{3}{2}}(u)\}.$$

B can now be found from (25) and the potential itself is then given by (24). If h is such that higher-order terms must be retained the algebra becomes somewhat complicated and it would probably be better to draw up tables of the L's and C's as numerical functions of h.

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NOTE ON DIFFRACTION BY AN EDGE

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SUMMARY

On the assumption that the electric current and charge on a perfectly conducting sheet are integrable in the neighbourhood of the edge, it is shown that the line distributions of electric current and charge on the edge are determined by the boundary conditions on the sheet. As a boundary condition on the edge it is postulated that there be no line distributions of magnetic current and charge; it is shown that the field is then determined uniquely. It is also proved that, if the currents near the edge can be expanded in power series, the index of the first term is $(2p+1)/2$, where p is an integer, greater than -2 or -1 according as the current parallel or perpendicular to the edge is considered. Finally the diffraction, in three dimensions, of a plane wave by a semi-infinite plane is obtained.

1. The expressions for the field

It has been pointed out by Bouwkamp (**1**) and Rayleigh (**2**) that the problem of diffraction by an edge is not solved uniquely unless some boundary condition on the edge is specified so that the singularities there can be determined. In what follows the singularities formed by electric current and charge, magnetic current and charge are discussed.

Let S be an unclosed surface bounded by a simple closed curve C which has neither double points nor discontinuous changes in direction. Let S_1 be the surface generated by the circumference of a circle of small radius δ, whose centre moves along C. Let S_2 be that part of S which is not inside S_1. We take S to be perfectly conducting and consider only periodic phenomena varying as $e^{i\omega t}$, this factor being omitted throughout. The m.k.s. system of units is used.

Let a field in which the electric intensity and magnetic flux density are $\mathbf{E}_0$, $\mathbf{B}_0$ respectively fall on S, which is surrounded by free space. Let $\mathbf{E}$, $\mathbf{B}$ be the electric intensity and magnetic flux density of the resulting field. Assuming that $\mathbf{E}-\mathbf{E}_0$, $\mathbf{B}-\mathbf{B}_0$ satisfy the radiation conditions at infinity* we can express $\mathbf{E}$, $\mathbf{B}$ in terms of their tangential components and their tangential derivatives on S_1+S_2 by means of the formulae

$$\mathbf{E} = \mathbf{E}_0 + \frac{i\omega}{4\pi k^2} \times$$

$$\times \oint_{S_1+S_2} \left\{ -k^2\psi \mathbf{B}_1 \wedge d\mathbf{S} + (d\mathbf{S}.\,\mathrm{curl}'\,\mathbf{B}_1)\mathrm{grad}'\,\psi + \frac{ik^2}{\omega}\,(d\mathbf{S}\wedge\mathbf{E}_1)\wedge\mathrm{grad}'\,\psi \right\},$$

* These conditions are not essential in sections 1–4 since $\mathbf{E}_0$, $\mathbf{B}_0$ can be replaced by any field which remains finite at the edge without affecting the argument.

$$\mathbf{B} = \mathbf{B}_0 + \frac{1}{4\pi} \oint_{S_1+S_2} \left\{ \frac{ik^2}{\omega} (\mathbf{E}_1 \wedge d\mathbf{S})\psi - (d\mathbf{S} \wedge \mathbf{B}_1) \wedge \text{grad}'\psi - (d\mathbf{S}.\mathbf{B}_1)\text{grad}'\psi \right\},$$

where $k = 2\pi/(\text{wave-length})$, $\psi = e^{-ikr}/r$, and

$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2,$$

(x, y, z) and (x', y', z') being the point of observation and a point of S_1 (or S_2) respectively (see Silver, **3**). The primes on the vector operators indicate operations with respect to (x', y', z'). On S_1, $\mathbf{E}_1$ and $\mathbf{B}_1$ are the electric intensity and magnetic flux density. On S_2, $\mathbf{n} \wedge \mathbf{E}_1$ and $\mathbf{n}.\mathbf{B}_1$ vanish, whereas in other terms $\mathbf{B}_1$ is the discontinuity of the magnetic flux density across S_2; $d\mathbf{S}$ $(= \mathbf{n}\, dS)$ is along the normal, out of the space under consideration, to S_1; on S_2, $d\mathbf{S}$ is along the normal, its direction being determined by the choice of $\mathbf{B}_1$. We suppose that the integral over S_2 tends to a limit as $\delta \to 0$, i.e. we take the electric current and charge on S_2 to be integrable in the neighbourhood of C. We shall in fact assume them to be Lebesgue-integrable. We further suppose that the integral over S_1 tends to a finite limit as $\delta \to 0$. This is certainly true when the fields are produced by line distributions of electric current and charge, magnetic current and charge on C. In general there will be distributions on C of electric current and charge included in the integral over S_2 when $\delta \to 0$. We consider the possible existence of line distributions (of finite magnitude) in excess of these.

2. The line distribution of electric current

If we consider the field of the line distribution at a point of S near C the dominant portion comes from the nearby part of C. From this point of view the most significant line distribution for the component of magnetic flux density normal to S is that of electric current since the nearby parts of the other line distributions effectively provide no contribution. Thus, supposing for the moment that only an extra line distribution of electric current is present, we have

$$\mathbf{B} = \mathbf{B}_0 - \frac{1}{4\pi} \int_S (d\mathbf{S} \wedge \mathbf{B}_1) \wedge \text{grad}'\psi + \text{curl} \oint_C I\psi\, d\mathbf{s},$$

where I is proportional to the strength of the line distribution and the integral along C is in the positive direction obtained by the right-hand screw law from $d\mathbf{S}$. On S, but not necessarily on C, the normal component of $\mathbf{B}$ vanishes and this condition supplies a certain integral equation.

Let O be a point on C and Ox that curve which is directed into S and which is the intersection of S and the plane normal to C at O. We consider the boundary condition at the point P, which is on Ox and is close to O

so that $OP\ (=x)$ is small. The contributions to $\mathbf{B}$ at P from $\mathbf{B}_0$ and those elements of S and C which are more than a certain small distance from O will be virtually independent of x and will be $o\,(1/x)$ as $x \to 0$. For the contribution of the near elements we shall obtain a sufficient approximation if we take x so small that the near elements of C can be regarded as lying on a straight line of length 2η with O as mid-point, while the near elements of S lie in a plane rectangle of sides 2η, ϵ. Here η, ϵ are small, but are chosen so that $\eta \gg \epsilon \gg x$. The y-axis at O is taken parallel to $\mathbf{n}$ and the z-axis is chosen so that the axes form a right-handed orthogonal system.

Let $\mathbf{B} \equiv (B_x, B_y, B_z)$, $\mathbf{B}_1 \equiv (B_{1x}, B_{1y}, B_{1z})$ and $I_x(x', z') = i\omega B_{1z}/4\pi k^2$, $I_z(x', z') = -i\omega B_{1x}/4\pi k^2$ so that I_x, I_z are proportional to the electric currents in the near elements of S. The contribution to B_y at P of the near elements is

$$\lim_{y\to 0} \frac{k^2}{i\omega} \int_0^{\epsilon} \int_{-\eta}^{\eta} \left\{ I_x(x', z') \frac{\partial \psi}{\partial z'} - I_z(x', z') \frac{\partial \psi}{\partial x'} \right\} dz'dx' = \frac{\partial}{\partial x} \int_{-\eta}^{\eta} I(z')\psi \, dz',$$

where $y' = 0$, $z = 0$ in ψ. We assume that $\partial I_x/\partial x'$, I_z are continuous and differentiable except possibly at $x' = 0$. We further assume that $\partial I_x/\partial x'$, I_z and their derivatives with respect to z' are Lebesgue-integrable in the neighbourhood of $x' = 0$. In general we are concerned with the possibility that $\partial I_x/\partial x'$, I_z may be unbounded at $x' = 0$; if they are bounded the analysis is considerably simpler.

In the range of integration under consideration we may take $\psi \approx 1/r$ and then

$$\int_{-\eta}^{\eta} I(z')\psi \, dz' = I(0) \log \frac{\eta + (\eta^2 + x^2)^{\frac{1}{2}}}{-\eta + (\eta^2 + x^2)^{\frac{1}{2}}} + O(1 + x^2).$$

Hence

$$\frac{\partial}{\partial x} \int_{-\eta}^{\eta} I(z')\psi \, dz' = -\frac{2I(0)}{x} + o\left(\frac{1}{x}\right).$$

Also

$$\begin{aligned} \lim_{y\to 0} \int_0^{\epsilon} \int_{-\eta}^{\eta} & \left\{ I_x(x', z') \frac{\partial \psi}{\partial z'} - I_z(x', z') \frac{\partial \psi}{\partial x'} \right\} dz'dx' \\ &= \lim_{y\to 0} \int_0^{\epsilon} \int_{-\eta}^{\eta} \left(-\psi \frac{\partial I_x}{\partial z'} - I_z \frac{\partial \psi}{\partial x'} \right) dz'dx' + o\left(\frac{1}{x}\right) \\ &= \lim_{y\to 0} \int_0^{\epsilon} \left(\frac{\partial I_x}{\partial z'} + I_z \frac{\partial}{\partial x'} \right) \log\{(x - x')^2 + y^2\} \, dx' + o\left(\frac{1}{x}\right), \end{aligned}$$

where

$$\frac{\partial I_x}{\partial z'} = \left[\frac{\partial}{\partial z'} I_x(x', z') \right]_{z'=0}, \qquad I_z = I_z(x', 0).$$

We can split the range of integration $[0, \epsilon]$ into the three intervals $[0, x/2]$, $[x/2, 3x/2]$, $[3x/2, \epsilon]$. In the first interval we put $y = 0$ and we have to consider integrals of the type $\int_0^{x/2} I_z \frac{dx'}{x-x'}$. Let $\phi(x) = \int_0^x |I_z|\, dx'$ so that $\phi(x)$ is absolutely continuous and $\phi(x) \to 0$ as $x \to 0$. Then

$$\left| \int_0^{x/2} I_z \frac{dx'}{x-x'} \right| \leqslant \int_0^{x/2} |I_z| \frac{dx'}{x-x'}$$

i.e.
$$< \frac{2}{x} \int_0^{x/2} |I_z|\, dx'$$

or
$$< \frac{2}{x} \phi\left(\frac{x}{2}\right) = o\left(\frac{1}{x}\right).$$

In the interval $[3x/2, \epsilon]$ we put $y = 0$ in the integrand and obtain an integral of the type $\int_{3x/2}^{\epsilon} I_z \frac{dx'}{x'-x}$. We have

$$\left| \int_{3x/2}^{\epsilon} I_z \frac{x\, dx'}{x'-x} \right| \leqslant \int_{3x/2}^{\epsilon} |I_z| \frac{x\, dx'}{x'-x},$$

i.e.
$$\leqslant \frac{x\,\phi(\epsilon)}{\epsilon - x} - 2\phi\left(\frac{3x}{2}\right) + \int_{3x/2}^{\epsilon} \phi(x') \frac{x\, dx'}{(x'-x)^2} \qquad (1)$$

by an integration by parts.

Now, given $\epsilon_1 > 0$, we can find X such that $\phi(x') < \epsilon_1$ for $0 \leqslant x' \leqslant X$ and we can choose x such that $x = X^2$. Then, providing $X < \frac{2}{3}$ and $X < \epsilon$,

$$\int_{3x/2}^{\epsilon} \phi(x') \frac{x\, dx'}{(x'-x)^2} < \int_{3x/2}^{X} \frac{\epsilon_1 x\, dx'}{(x'-x)^2} + \int_{X}^{\epsilon} \frac{\phi(x') x\, dx'}{(x'-x)^2},$$

i.e.
$$< 2\epsilon_1 + \frac{(M-\epsilon_1)x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}} - \frac{Mx}{\epsilon - x},$$

where M is the upper bound of $\phi(x')$ in $[X, \epsilon]$. It now follows that as $x \to 0$ we can make the right-hand side of (1) as small as we please by a suitable choice of ϵ_1. Hence

$$\int_{3x/2}^{\epsilon} I_z \frac{dx'}{x'-x} = o\left(\frac{1}{x}\right).$$

Finally we consider

$$\left|\int_{x/2}^{3x/2} xI_z \frac{x-x'}{(x-x')^2+y^2}\,dx'\right| \leqslant \int_{x/2}^{x} x|I_z| \frac{x-x'}{(x-x')^2+y^2}\,dx' + \int_{x}^{3x/2} x|I_z| \frac{x'-x}{(x-x')^2+y^2}\,dx'$$

i.e. $$\leqslant \frac{2x^2\{\phi(3x/2)-\phi(x/2)\}}{x^2+4y^2} - \left(\int_{x/2}^{x} - \int_{x}^{3x/2}\right)\left\{x\phi(x')\,\frac{\partial}{\partial x'}\,\frac{x-x'}{(x-x')^2+y^2}\,dx'\right\}$$

by an integration by parts. Now

$$\int_{x/2}^{x} x\,\phi(x')\,\frac{\partial}{\partial x'}\,\frac{x-x'}{(x-x')^2+y^2}\,dx' = -\frac{2x^2}{x^2+4y^2}\,\{\phi(x)+o(1)\}$$

since $\phi(x')-\phi(x) = o(1)$ as $x \to 0$, with x' in $[x/2, x]$. Similarly for the other integral. Hence

$$\lim_{y\to 0} \int_{x/2}^{3x/2} I_z \frac{x-x'}{(x-x')^2+y^2}\,dx' = o\left(\frac{1}{x}\right),$$

and thus $$\lim_{y\to 0} \int_{0}^{\epsilon} I_z \frac{\partial}{\partial x'} \log\{(x-x')^2+y^2\}\,dx' = o\left(\frac{1}{x}\right).$$

Also $$\lim_{y\to 0} \int_{0}^{\epsilon} \frac{\partial I_x}{\partial z'} \log\{(x-x')^2+y^2\}\,dx' = \int_{0}^{\epsilon} \frac{\partial I_x}{\partial z'} \log|x-x'|\,dx' = o\left(\frac{1}{x}\right).$$

Consequently we have shown that

$$B_y = \frac{2I(0)}{x} + o\left(\frac{1}{x}\right).$$

Thus in order that the boundary condition $B_y = 0$ can be satisfied for small x we must have $I(0) = 0$, i.e. there is no excess electric current on C.

Of the other possible excess line distributions mentioned, none provides a contribution from the near elements to B_y and so our conclusion remains valid even when the other singularities are present.

3. The other line distributions

We now suppose that the current perpendicular to C is non-zero so that a line of electric charge accumulates on C. The contribution of the near elements to E_x is then

$$\lim_{y\to 0} \int_{0}^{\epsilon} \int_{-\eta}^{\eta} \left[k^2\psi I_x(x', z') - \left\{\frac{\partial}{\partial x'}\,I_x(x', z') + \frac{\partial}{\partial z'}\,I_z(x', z')\right\} \frac{\partial\psi}{\partial x'}\right] dz'dx' + \\ + \frac{\partial}{\partial x} \int_{-\eta}^{\eta} I_x(0, z')\psi\,dz'.$$

The dominant portion due to the electric charge is $-2I_x(0,0)/x$. The integral involving $\partial I_x/\partial x' + \partial I_z/\partial z'$ is of the type discussed in connexion with B_y and is consequently $o(1/x)$. The integral involving I_x is clearly $o(1/x)$. Hence E_x can vanish for small x only if $I_x(0,0) = 0$, i.e. there is no line of charge on C and the current perpendicular to C vanishes at C.

The remaining two line distributions, those of magnetic current and charge, provide no contribution to E_x, E_z, or B_y from the near elements and so they cannot be determined from the existing boundary conditions. The surface of a perfect conductor cannot support either magnetic charge or current and so a reasonable and simple assumption is that the edge cannot support such distributions. This condition is equivalent to certain restrictions on the field.

4. Behaviour of the field near the edge*

We now introduce the cylindrical polar coordinates (ρ, ϕ, z),† keeping the origin at O. The components E_ρ, E_z, B_ϕ are unaffected by the presence of magnetic current or charge on the edge. Hence the contribution of the near elements to B_ϕ is

$$\frac{k^2}{i\omega}\int_0^\epsilon\int_{-\eta}^{\eta}\left\{I_x(x',z')\,\frac{\partial\psi}{\partial z'}\cos\phi + I_z(x',z')\,\frac{\partial\psi}{\partial\rho}\right\}dz'dx', \tag{2}$$

where, in ψ, $r^2 = \rho^2 + x'^2 - 2x'\rho\cos\phi + z'^2$. The integral involving I_z has already been discussed in connexion with B_y for the cases $\phi = 0$, 2π and shown to be $o(1/\rho)$ as $\rho \to 0$ in those cases. The first integral is known to be finite from potential theory. Hence $B_\phi = o(1/\rho)$, and similarly we may show $E_\rho = o(1/\rho)$.

The contribution to E_z from the near elements is

$$\int_0^\epsilon\int_{-\eta}^{\eta}\left[k^2 I_z(x',z')\psi - \left\{\frac{\partial}{\partial x'}\,I_x(x',z') + \frac{\partial}{\partial z'}\,I_z(x',z')\right\}\frac{\partial\psi}{\partial z'}\right]dz'dx'.$$

It is not difficult to show that this expression is continuous across $x = 0$. In other words, E_z is continuous across $x = 0$. Also since $E_z = 0$ for all x satisfying $0 < x < \epsilon$ it follows that $E_z = 0$ at $x = 0$, i.e. the component of electric intensity along C vanishes on C. Hence $E_z = o(1)$ in the vicinity of the edge.

Collecting these results together we have

$$E_z = o(1), \qquad E_\rho = o(1/\rho), \qquad B_\phi = o(1/\rho). \tag{3}$$

* *Note added in proof.* For a recent discussion, from a different point of view, of the behaviour of the field near the edge see J. Meixner, *Ann. der Phys.* **6** (1949), 2.

† In this section ϕ has a different meaning from that in earlier sections.

When there are no line distributions of magnetic current and charge present we may easily show, by considering the contribution of the near elements, that

$$B_z = \text{const.} + o(1), \qquad E_\phi = o(1/\rho), \qquad B_\rho = o(1/\rho). \tag{4}$$

If magnetic current and charge were present there would be a term of $O(\log \rho)$ in B_z and terms of $O(1/\rho)$ in E_ϕ, B_ρ so that the absence of these singularities allows the behaviour in (4). Conversely, if we assume the behaviour in (4) the condition on E_ϕ ensures that no magnetic current is present and then the condition on B_ρ requires that there shall be no magnetic charge.

5. Uniqueness

We can now show that the field is determined uniquely by (3), (4), the conditions on the surface, and the usual radiation conditions at infinity. Suppose that two solutions are possible. Let $\mathbf{E}$, $\mathbf{B}$ represent the difference between them. The behaviour of $\mathbf{E}$, $\mathbf{B}$ near the edge will be that indicated in the previous section. Let S_3 be the surface of a sphere of radius R and centre in the neighbourhood of C, where R is taken large enough for S_3 to enclose S completely. Then

$$\oint_{S_1+S_2+S_3} (\mathbf{E} \wedge \mathbf{B}^* + \mathbf{E}^* \wedge \mathbf{B}).d\mathbf{S} = 0,$$

where $\mathbf{E}^*$, $\mathbf{B}^*$ are the complex conjugates of $\mathbf{E}$, $\mathbf{B}$ respectively. Now

$$\begin{aligned}\int_{S_1} \mathbf{E} \wedge \mathbf{B}^*.d\mathbf{S} &= \oint_C \int_0^{2\pi} (E_\phi B_z^* - E_z B_\phi^*)\delta \, d\phi ds \\ &= o(1)\end{aligned}$$

as $\delta \to 0$, from (3) and (4). Also

$$\int_{S_2} (\mathbf{E} \wedge \mathbf{B}^* + \mathbf{E}^* \wedge \mathbf{B}).d\mathbf{S} = 0$$

since the tangential components of $\mathbf{E}$ vanish on S_2. Hence allowing $\delta \to 0$ we have

$$\oint_{S_3} (\mathbf{E} \wedge \mathbf{B}^* + \mathbf{E}^* \wedge \mathbf{B}).d\mathbf{S} = 0, \tag{5}$$

i.e. the integral of the Poynting vector over any closed surface enclosing S is zero. Consequently we may expect on physical grounds that $\mathbf{E} \equiv 0$, $\mathbf{B} \equiv 0$. We can show this mathematically by recalling that since $\mathbf{E}$, $\mathbf{B}$ satisfy the radiation condition at infinity we can write, using spherical

polar coordinates R, θ, ϕ,

$$\left.\begin{aligned}\mathbf{E} &= e^{-ikR}\sum_{n=1}\left(\frac{a_{n1}}{R^{n+1}}\,\mathbf{i}_1+\frac{a_{n2}}{R^n}\,\mathbf{i}_2+\frac{a_{n3}}{R^n}\,\mathbf{i}_3\right),\\ \mathbf{B} &= e^{-ikR}\sum_{n=1}\left(\frac{b_{n1}}{R^{n+1}}\,\mathbf{i}_1+\frac{b_{n2}}{R^n}\,\mathbf{i}_2+\frac{b_{n3}}{R^n}\,\mathbf{i}_3\right),\end{aligned}\right\} \tag{6}$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are unit vectors in the directions R, θ, ϕ increasing respectively. The a's and b's depend only on θ and ϕ, and the series are absolutely and uniformly convergent for $R >$ some constant (R_0, say). Also we know that

$$a_{12} = \frac{\omega}{k}\,b_{13}, \qquad a_{13} = -\frac{\omega}{k}\,b_{12} \tag{7}$$

and that if $a_{12} = 0$ and $a_{13} = 0$ then $\mathbf{E} \equiv 0$, $\mathbf{B} \equiv 0$. From (5) and (6) we have for $R > R_0$,

$$\oint_{S_3}\left\{\sum_{n=1}\frac{a_{n2}}{R^n}\sum_{n=1}\frac{b^*_{n3}}{R^n}-\sum_{n=1}\frac{a_{n3}}{R^n}\sum_{n=1}\frac{b^*_{n2}}{R^n}+ \right.$$
$$\left.+\sum_{n=1}\frac{a^*_{n2}}{R^n}\sum_{n=1}\frac{b_{n3}}{R^n}-\sum_{n=1}\frac{a^*_{n3}}{R^n}\sum_{n=1}\frac{b_{n2}}{R^n}\right\}R^2\,d\Omega = 0$$

where $R^2\,d\Omega = dS$. Hence, using (7),

$$\oint_{S_3}(a_{12}a^*_{12}+a_{13}a^*_{13})\,d\Omega = 0.$$

Since this is true for all S_3 such that $R > R_0$ we must have $a_{12} = 0$, $a_{13} = 0$ for $R > R_0$, and hence $\mathbf{E} \equiv 0$, $\mathbf{B} \equiv 0$. Hence the field is determined uniquely.

6. A particular form for the current

In this section we assume a special form for the current which satisfies our previous requirements, and investigate the restrictions imposed by the boundary conditions on the sheet. Suppose that near the edge we can expand I_x, I_z in the series

$$I_x(x', z') = x'^a\{f_1(z')+x'f_2(z')+\ldots\},$$
$$I_z(x', z') = x'^b\{g_1(z')+x'g_2(z')+\ldots\},$$

where $a > 0$, $b > -1$, and $f_1, f_2, \ldots, g_1, g_2, \ldots$ are continuous functions of z' possessing continuous derivatives. We obtain from (2) that the contribution of the near elements to B_ϕ is

$$\frac{k^2}{i\omega}\int_0^\epsilon\left\{x'^a f'_1(0)\cos\phi - x'^b g_1(0)\,\frac{\partial}{\partial\rho}\right\}\log(x'^2-2x'\rho\cos\phi+\rho^2)\,dx'+(\ldots), \tag{8}$$

where $f'_1(0) = [df/dz']_{z'=0}$ and the dots in brackets represent terms whose form will be described presently.

Now

$$\int_0^{\epsilon} w^a \log(w^2-2w\rho\cos\phi+\rho^2)\,dw$$
$$=\frac{\epsilon^{a+1}}{a+1}\log(\epsilon^2-2\epsilon\rho\cos\phi+\rho^2)-\frac{2}{a+1}\int_0^{\epsilon}\frac{w^{a+1}(w-\rho\cos\phi)}{w^2-2w\rho\cos\phi+\rho^2}\,dw \qquad (8a)$$

by an integration by parts. Regard w as the complex variable $|w|e^{i\theta}$ and define w^{a+1} by $w^{a+1}=|w|^{a+1}e^{i(a+1)\theta}$ $(0\leqslant\theta<2\pi)$. Let Γ be the contour defined by the circles $|w|=\epsilon$, $|w|=\delta'$ $(0\leqslant\theta\leqslant 2\pi,\ \delta'\ll\rho\ll\epsilon)$ and the straight lines $\theta=0$, 2π $(\delta'\leqslant|w|\leqslant\epsilon)$. Then

$$\int_{\Gamma}\frac{w^{a+1}(w-\rho\cos\phi)\,dw}{w^2-2w\rho\cos\phi+\rho^2}=2\pi i e^{i(a+1)\pi}\rho^{a+1}\cos\{(a+1)(\phi-\pi)\}.$$

The integral over $|w|=\delta'$ vanishes as $\delta'\to 0$ and hence, when a is not an integer,

$$\int_0^{\epsilon}\frac{w^{a+1}(w-\rho\cos\phi)\,dw}{w^2-2w\rho\cos\phi+\rho^2}$$
$$=\frac{-\pi\rho^{a+1}\cos\{(a+1)(\phi-\pi)\}}{\sin(a+1)\pi}+\frac{e^{-i(a+1)\pi}\epsilon^{a+2}}{2\sin(a+1)\pi}\int_0^{2\pi}\frac{(\epsilon e^{i\theta}-\rho\cos\phi)e^{i(a+2)\theta}\,d\theta}{\epsilon^2e^{2i\theta}-2\epsilon\rho e^{i\theta}\cos\phi+\rho^2}$$
$$=\frac{-\pi\rho^{a+1}\cos\{(a+1)(\phi-\pi)\}}{\sin(a+1)\pi}+\text{non-negative integral powers of }\rho.$$

We may proceed similarly for the term involving $g_1(0)$ in (8) when b is not an integer, and hence obtain for the contribution from the near elements to B_ϕ

$$\frac{2\pi k^2 f_1'(0)\rho^{a+1}\cos\{(a+1)(\phi-\pi)\}}{i\omega(a+1)\sin(a+1)\pi}\cos\phi-$$
$$-\frac{2\pi k^2 g_1(0)\rho^b\cos\{(b+1)(\phi-\pi)\}}{i\omega\sin(b+1)\pi}+(\ldots), \qquad (9)$$

where the dots in brackets, like those in (8), indicate terms whose dependence on ρ is of the type ρ^{a+n+2}, ρ^{b+n+1} or ρ^n, where n is zero or any positive integer.

These bracketed terms of (9) include those of (8). The contribution of the far elements and of the incident field can also be included in the dots of (9) and so (9) represents B_ϕ near the edge.

In order that $B_\phi=0$ when $\phi=0$, 2π the coefficients of the powers of ρ must vanish. Since neither a nor b is an integer, (9) shows that, if $b\neq a+1$, both a and b are odd half integers. If $b=a+1$ we can obtain no further

information about a and b from (9) but the expression for E_ρ is of the same form as (9) together with an additional term $\rho^{a-1}\dfrac{\cos\{a(\phi-\pi)\}}{\sin a\pi}$ and thus a and b are both odd half integers.

If a is an integer, the integral on the right-hand side of (8 a) can be evaluated by elementary means to give a series of non-negative powers of ρ plus a term $\rho^{a+1}\cos(a+1)\phi\log\rho$. Making the appropriate modification to (9) the condition on B_ϕ cannot be satisfied unless both a and b are integers such that $b = a+1$. But then the condition on E_ρ cannot be satisfied. Hence all permissible values of a, b are given by the scheme

$$a = (2p+1)/2, \qquad b = (2q+1)/2, \tag{10}$$

where p, q are integers such that $p \geqslant 0$, $q \geqslant -1$.

It has been tacitly assumed in the above that neither $f_1'(0) = 0$ nor $g_1(0) = 0$; this assumption is valid in the neighbourhood of some point of C unless one or other (or both) of the currents vanishes throughout a strip of width ϵ with C as edge. The behaviour of the non-zero current will be that found above.

It is to be noted that, if in section 2 we had made the assumptions on the current used here, (9) and (10) would imply that there is no excess line distribution of electric current on C; the form of E_ρ then implies that there is no excess electric charge.

7. Plane wave incident on a semi-infinite screen

The analysis in the previous sections has been developed for the case where C is closed, but it is clear that it applies equally well if C extends to infinity as, for example, on a semi-infinite plane, providing that slight modifications are made to S_2 and S_3.

We now consider the diffraction of a harmonic plane wave by a perfectly conducting semi-infinite plane. Wiegrefe (**4**) has indicated that a solution of this problem can be obtained by splitting the total field into two partial fields: (a) that which has no component of the electric vector normal to the plane, and (b) that which has no magnetic component normal to the plane. It can be seen at once that the solution for (a) does not satisfy our boundary conditions at the edge since $E_z = O(\rho^{-\frac{1}{2}})$. However, this case will not arise frequently since it can occur only when there is no electric charge on the plane. We shall obtain a solution satisfying our boundary conditions by means of integral equations.

Let the edge of the plane be the z-axis and let the y-axis be normal to the plane so that the plane is defined by $y = 0$, $x \geqslant 0$. The direction of the incident wave normal is specified by the spherical polar angles θ_1, ϕ_1

where $\frac{1}{2}\pi \geqslant \theta_1 > 0$, $\pi > \phi_1 > 0$. The electric intensity of the plane wave is given by $(\lambda, \mu, \nu)\exp(-\alpha x - \beta y - \gamma z)$, where

$$\alpha = ik\sin\theta_1\cos\phi_1, \qquad \beta = ik\sin\theta_1\sin\phi_1, \qquad \gamma = ik\cos\theta_1.$$

The constants λ, μ, ν are to be chosen suitably so that the wave is plane. Let B_{1x}, B_{1z} be the discontinuities in the x- and z-components of $\mathbf{B}$ across the conducting sheet and then, in view of the foregoing analysis, the electric intensity is given by

$$\mathbf{E} = (\lambda\mathbf{i}+\mu\mathbf{j}+\nu\mathbf{k})\exp(-\alpha x-\beta y-\gamma z)+$$
$$+\frac{i\omega}{4\pi k^2}\int_0^\infty\int_{-\infty}^\infty\left[k^2\psi\{B_{1z}(x',z')\mathbf{i}-B_{1x}(x',z')\mathbf{k}\}+\right.$$
$$\left.+\left\{\frac{\partial}{\partial z'}B_{1x}(x',z')-\frac{\partial}{\partial x'}B_{1z}(x',z')\right\}\mathrm{grad}'\,\psi\right]dz'dx',$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in the x, y, z directions respectively and in r we put $y' = 0$ after the operations have been performed. B_{1x}, B_{1z} are functions of x', z', but it is clear physically that their dependence on z' must be through $e^{-\gamma z'}$ and that only. Hence we may write

$$B_{1z} = \frac{4\pi k^2}{i\omega}J_x(x')e^{-\gamma z'}, \qquad B_{1x} = -\frac{4\pi k^2}{i\omega}J_z(x')e^{-\gamma z'},$$

where J_x, J_z are functions of x' only. We thus find, using

$$\int_{-\infty}^{\infty}\psi e^{-\gamma z'}\,dz' = -\pi i e^{-\gamma z}H_0^{(2)}[\kappa\{(x-x')^2+(y-y')^2\}^{\frac{1}{2}}],$$

that

$$\left.\begin{aligned}
E_x e^{\gamma z} &= \lambda e^{-\alpha x-\beta y}-i\pi\int_0^\infty\left\{k^2J_x+\left(\gamma J_z-\frac{\partial J_x}{\partial x'}\right)\frac{\partial}{\partial x'}\right\}H_0^{(2)}[\kappa\{(x-x')^2+y^2\}^{\frac{1}{2}}]\,dx',\\
E_y e^{\gamma z} &= \mu e^{-\alpha x-\beta y}-i\pi\int_0^\infty\left(\gamma J_z-\frac{\partial J_x}{\partial x'}\right)\frac{\partial}{\partial y'}H_0^{(2)}[\kappa\{(x-x')^2+(y-y')^2\}^{\frac{1}{2}}]\,dx',\\
E_z e^{\gamma z} &= \nu e^{-\alpha x-\beta y}-i\pi\int_0^\infty\left(\kappa^2J_z-\gamma\frac{\partial J_x}{\partial x'}\right)H_0^{(2)}[\kappa\{(x-x')^2+y^2\}^{\frac{1}{2}}]\,dx'
\end{aligned}\right\} \tag{11}$$

where $H_0^{(2)}(x)$ is the Hankel function of the second kind and

$$\kappa = (\gamma^2+k^2)^{\frac{1}{2}} = k\sin\theta_1.$$

The vanishing of E_x, E_z on the sheet now gives the following integral equations to determine J_x, J_z:

$$\int_0^\infty \left(\kappa^2 - \gamma J_x \frac{\partial J_x}{\partial x'}\right) H_0^{(2)}(\kappa|x-x'|)\, dx' - \frac{\nu}{\pi i} e^{-\alpha x} = 0, \tag{12}$$

$$\lim_{y\to 0} \int_0^\infty \left\{k^2 J_x + \left(\gamma J_z - \frac{\partial J_x}{\partial x'}\right)\frac{\partial}{\partial x'}\right\} H_0^{(2)}\left[\kappa\{(x-x')^2+y^2\}^{\frac{1}{2}}\right] dx' - \frac{\lambda}{\pi i} e^{-\alpha x} = 0, \tag{13}$$

the equations holding for $x > 0$.

It is now convenient to assume that k has a small negative imaginary part so that $k = k_r - ik_i$ ($k_r > 0$, $k_i > 0$). We search for solutions of (12), (13) such that $J_x(x') = 0$ with $x' \leqslant 0$, $J_z(x') = 0$ with $x' < 0$ and such that $J_x(x') = O(e^{-\alpha x'})$, $J_z(x') = O(e^{-\alpha x'})$ for large positive x'.

Equations (12), (13) are simultaneous integral equations of the first kind of the Wiener–Hopf type. The technique for solving single equations of this type is well known and a slight adaptation of it solves our equations. It is, of course, simple to transform (12) and (13) into two single equations, but the following procedure is adopted since it will apply to the solution of simultaneous equations where the transformation is less simple. Each equation is extended to hold for all x by adding to the right-hand side an unknown function which is zero for $x > 0$ and equal to the integral for $x < 0$; the term involving $e^{-\alpha x}$ is taken to be zero for $x < 0$. Then each of the unknown functions is $O(e^{i\kappa x})$ as $x \to -\infty$. We now take the bilateral Laplace transform with respect to x, i.e. $\int_{-\infty}^{\infty} e^{-sx}$, and assuming that the limit and transforming integration can be inverted we obtain

$$\frac{\kappa^2 \mathscr{J}_z - \gamma s \mathscr{J}_x}{(s^2+\kappa^2)^{\frac{1}{2}}} - \frac{\nu}{2\pi i}\frac{1}{s+\alpha} = \psi_1(s), \tag{14}$$

$$\frac{-\gamma s \mathscr{J}_z + (s^2+k^2)\mathscr{J}_x}{(s^2+\kappa^2)^{\frac{1}{2}}} - \frac{\lambda}{2\pi i}\frac{1}{s+\alpha} = \psi_2(s), \tag{15}$$

where $\mathscr{J}_x = \int_0^\infty J_x(x)e^{-sx}\, dx$, $\mathscr{J}_z = \int_0^\infty J_z(x)e^{-sx}\, dx$ and that branch of $(s^2+\kappa^2)^{\frac{1}{2}}$ is chosen which reduces to κ when $s = 0$. Since $k_i > 0$, $\mathrm{im}(\kappa) < 0$ and the transform of the Hankel function is analytic in the strip $\mathrm{im}(\kappa) < \sigma < -\mathrm{im}(\kappa)$ ($s = \sigma + i\tau$). $\mathscr{J}_x$, $\mathscr{J}_z$ and $1/(s+\alpha)$ are analytic bounded functions in $\sigma > -\mathrm{re}(\alpha)$. $\psi_1(s)$, $\psi_2(s)$ are unknown analytic bounded functions in $\sigma < \mathrm{re}(\alpha)$ (or $-\mathrm{im}(\kappa)$). Hence (14), (15) hold in the strip

$$-\mathrm{re}(\alpha) < \sigma < \mathrm{re}(\alpha)$$

since $\mathrm{re}(\alpha) < -\mathrm{im}(\kappa)$.

Eliminating $\mathscr{J}_z$ from (14), (15) we have

$$k^2(s^2+\kappa^2)^{\frac{1}{2}}\mathscr{J}_x-\frac{\gamma\nu s+\lambda\kappa^2}{2\pi i(s+\alpha)}=\gamma s\psi_1(s)+\kappa^2\psi_2(s),$$

or

$$k^2(s+i\kappa)^{\frac{1}{2}}\mathscr{J}_x-\frac{\gamma\nu s+\lambda\kappa^2}{2\pi i(s+\alpha)(-\alpha-i\kappa)^{\frac{1}{2}}}$$

$$=\frac{\gamma s\psi_1(s)+\kappa^2\psi_2(s)}{(s-i\kappa)^{\frac{1}{2}}}+\frac{\gamma\nu s+\lambda\kappa^2}{2\pi i(s+\alpha)}\left\{\frac{1}{(s-i\kappa)^{\frac{1}{2}}}-\frac{1}{(-\alpha-i\kappa)^{\frac{1}{2}}}\right\}. \quad (16)$$

In (16) the left-hand side is analytic in $\sigma > -\mathrm{re}(\alpha)$, whereas the right-hand side is analytic in $\sigma < \mathrm{re}(\alpha)$. Since the expressions have a strip in common it follows that both sides are equal to an integral function. Further, as $|s| \to \infty$ with $\sigma > -\mathrm{re}(\alpha)$, the left-hand side is $o(s^{\frac{1}{2}})$ since $\mathscr{J}_x \to 0$ as $|s| \to \infty$; and as $|s| \to \infty$, with $\sigma < \mathrm{re}(\alpha)$, the right-hand side is $o(s^{\frac{1}{2}})$ since $\psi_1(s) \to 0$, $\psi_2(s) \to 0$ as $|s| \to \infty$. It follows, by the extension of Liouville's theorem, that the integral function is a constant A.

Hence
$$\mathscr{J}_x=\frac{A}{k^2(s+i\kappa)^{\frac{1}{2}}}+\frac{\gamma\nu s+\lambda\kappa^2}{2\pi ik^2(-\alpha-i\kappa)^{\frac{1}{2}}(s+\alpha)(s+i\kappa)^{\frac{1}{2}}} \quad (17)$$

and
$$\frac{\gamma s\psi_1(s)+\kappa^2\psi_2(s)}{(s-i\kappa)^{\frac{1}{2}}}+\frac{\gamma\nu s+\lambda\kappa^2}{2\pi i(s+\alpha)}\left\{\frac{1}{(s-i\kappa)^{\frac{1}{2}}}-\frac{1}{(-\alpha-i\kappa)^{\frac{1}{2}}}\right\}=A. \quad (18)$$

But $J_x(x) \to 0$ as $x \to 0$ and hence in (17)

$$A+\frac{\nu\gamma}{2\pi i(-\alpha-i\kappa)^{\frac{1}{2}}}=0.$$

Therefore
$$\mathscr{J}_x=\frac{\lambda\kappa^2-\nu\alpha\gamma}{2\pi ik^2(-\alpha-i\kappa)^{\frac{1}{2}}(s+\alpha)(s+i\kappa)^{\frac{1}{2}}}. \quad (19)$$

Using the value of A in (18) we see that as $|s| \to \infty$, with $\sigma < \mathrm{re}(\alpha)$, $\psi_1(s)$ is $o(1/s)$ or $O(1/s)$ according as ν is or is not zero. This was to be expected since $\psi_1(s)$ is the transform, over $[-\infty, 0]$, of $E_z - E_{z0}$ which is $o(1)$ or $O(1)$ as $x \to 0$ according as ν is or is not zero.

We now return to equation (14) and observe that it can be written

$$\frac{\kappa^2\mathscr{J}_z-\gamma s\mathscr{J}_x}{(s+i\kappa)^{\frac{1}{2}}}-\frac{\nu(-\alpha-i\kappa)^{\frac{1}{2}}}{2\pi i(s+\alpha)}=(s-i\kappa)^{\frac{1}{2}}\psi_1(s)+\frac{\nu}{2\pi i(s+\alpha)}\{(s-i\kappa)^{\frac{1}{2}}-(-\alpha-i\kappa)^{\frac{1}{2}}\}.$$

The argument proceeds as for (16) and both sides are equal to an integral function; using (19) and our information on the behaviour of $\psi_1(s)$ at infinity, we find that the integral function is zero. Hence

$$\mathscr{J}_z=\frac{\gamma s(\lambda\kappa^2-\nu\alpha\gamma)}{2\pi ik^2\kappa^2(-\alpha-i\kappa)^{\frac{1}{2}}(s+\alpha)(s+i\kappa)^{\frac{1}{2}}}+\frac{\nu(-\alpha-i\kappa)^{\frac{1}{2}}(s+i\kappa)^{\frac{1}{2}}}{2\pi i\kappa^2(s+\alpha)}. \quad (20)$$

The application of the Mellin inversion formula to (19), (20) gives for J_x, J_z

$$J_x = \frac{\lambda\kappa^2 - \nu\alpha\gamma}{2\pi^{\frac{3}{2}} ik^2(-\alpha - i\kappa)^{\frac{1}{2}}} e^{-\alpha x} \int_0^x \frac{e^{(\alpha - i\kappa)u}}{u^{\frac{1}{2}}}\, du, \tag{21}$$

$$J_z = \frac{i}{2\pi^{\frac{3}{2}} k^2 \kappa(-\alpha - i\kappa)^{\frac{1}{2}}} \times \times \left[\{(\alpha\kappa + ik^2)\nu - \gamma\kappa\lambda\} \frac{e^{-i\gamma x}}{x^{\frac{1}{2}}} + \kappa\{\alpha\gamma\lambda - \nu(\alpha^2 + k^2)\} e^{-\alpha x} \int_0^x \frac{e^{(\alpha - i\kappa)u}}{u^{\frac{1}{2}}}\, du \right]. \tag{22}$$

In these formulae we may now take k to be real. In the particular case when $\theta_1 = \frac{1}{2}\pi$, (21), (22) reduce to the usual formulae for two dimensions.

The electric intensity at any point can be obtained by substituting (21), (22) in (11) but it is simpler to write (11) in the Mellin inversion form, e.g.

$$E_z e^{\gamma z} = \nu e^{-\alpha x - \beta y} - \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{\kappa^2 \mathscr{J}_z - \gamma s \mathscr{J}_x}{(s^2 + \kappa^2)^{\frac{1}{2}}} \exp\{sx - i|y|(s^2 + \kappa^2)^{\frac{1}{2}}\}\, ds,$$

where $-\mathrm{re}(\alpha) < \sigma_1 < \mathrm{re}(\alpha)$. Substituting from (19), (20)

$$E_x e^{\gamma z} = \lambda e^{-\alpha x - \beta y} + \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \left\{ \frac{(\nu\alpha\gamma - \lambda\kappa^2)(s - i\kappa)^{\frac{1}{2}}}{\kappa^2(-\alpha - i\kappa)^{\frac{1}{2}}(s + \alpha)} + \frac{\gamma\nu s(-\alpha - i\kappa)^{\frac{1}{2}}}{\kappa^2(s + \alpha)(s - i\kappa)^{\frac{1}{2}}} \right\} \times \times \exp\{sx - i|y|(s^2 + \kappa^2)^{\frac{1}{2}}\}\, ds,$$

$$E_y e^{\gamma z} = \mu e^{-\alpha x - \beta y} - \frac{\mathrm{sgn}\, y}{2\pi} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \left\{ \frac{(\nu\alpha\gamma - \lambda\kappa^2)s}{\kappa^2(-\alpha - i\kappa)^{\frac{1}{2}}(s + \alpha)(s + i\kappa)^{\frac{1}{2}}} + \right. + \left. \frac{\gamma\nu(-\alpha - i\kappa)^{\frac{1}{2}}(s + i\kappa)^{\frac{1}{2}}}{\kappa^2(s + \alpha)} \right\} \exp\{sx - i|y|(s^2 + \kappa^2)^{\frac{1}{2}}\}\, ds,$$

$$E_z e^{\gamma z} = \nu e^{-\alpha x - \beta y} - \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{\nu(-\alpha - i\kappa)^{\frac{1}{2}}}{(s + \alpha)(s - i\kappa)^{\frac{1}{2}}} \exp\{sx - i|y|(s^2 + \kappa^2)^{\frac{1}{2}}\}\, ds,$$

where $\mathrm{sgn}\, y = \pm 1$ according as $y \gtrless 0$.

It is to be noted that since the integrals for E_x, E_z are uniformly convergent with respect to y we may put $y = 0$ in the integrand to obtain values on the sheet. We see, therefore, that the inversion referred to earlier is permissible.

The behaviour of the electric intensity at large distances from the edge can be obtained by means of the saddle-point method of approximation. Let $x = R\cos\phi$, $y = R\sin\phi$ $(0 \leqslant \phi \leqslant 2\pi)$, where R is large. Then the saddle point of $sx - i|y|(s^2 + \kappa^2)^{\frac{1}{2}}$ occurs at $s = -i\kappa\cos\phi$. In order to

deform the contour so as to pass through the saddle point it is necessary to pass over the pole $s = -\alpha$ if $\phi_1 > \phi \geqslant 0$ or $2\pi \geqslant \phi > 2\pi - \phi_1$ but not otherwise. The region corresponding to $\phi_1 > \phi$ is the shadow and in this region the contribution from the pole removes the incident field. Corresponding to $2\pi \geqslant \phi > 2\pi - \phi_1$ is the region of reflection and here the pole provides the reflected wave. The diffracted field is supplied by the saddle point, and for this portion of the field we obtain

$$E_x e^{\gamma z} \sim \frac{\nu\alpha\gamma - \lambda\kappa^2 - \kappa(\lambda\kappa + i\nu\gamma)\cos\phi}{(-\alpha - i\kappa)^{\frac{1}{2}}(\alpha - i\kappa\cos\phi)\kappa\pi^{\frac{1}{2}}} \sin\tfrac{1}{2}\phi \, \frac{e^{-ikR}}{R^{\frac{1}{2}}},$$

$$E_y e^{\gamma z} \sim -\frac{\nu\gamma(\alpha + i\kappa) - \kappa(\lambda\kappa + i\nu\gamma)\cos\phi}{\pi^{\frac{1}{2}}\kappa(-\alpha - i\kappa)^{\frac{1}{2}}(\alpha - i\kappa\cos\phi)} \, |\cos\tfrac{1}{2}\phi| \operatorname{sgn} y \, \frac{e^{-ikR}}{R^{\frac{1}{2}}},$$

$$E_z e^{\gamma z} \sim \frac{\nu(-\alpha - i\kappa)^{\frac{1}{2}} \sin\tfrac{1}{2}\phi}{i\pi^{\frac{1}{2}}(\alpha - i\kappa\cos\phi)} \, \frac{e^{-ikR}}{R^{\frac{1}{2}}}.$$

except for values of ϕ near ϕ_1 and $2\pi - \phi_1$, for then the saddle point is in the vicinity of the pole $s = -\alpha$.

I wish to express my appreciation of some helpful comments by Professor S. Goldstein.

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NON-STEADY FLOWS UNDER ASYMPTOTIC SUCTION CONDITIONS

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SUMMARY

In a previous paper (**1**) steady state solutions were obtained for the circulatory flow about a circular cylinder, through the surface of which uniform suction is applied.

In this paper approximate solutions are obtained in a closed form for the transient state occurring when the peripheral speed of the cylinder and/or the suction velocity through the surface is suddenly altered. The suction velocity is assumed to be sufficiently large for the asymptotic exponential velocity distribution to be closely approached in the initial and final steady states.

The velocity distributions in the transient state lie between those for the initial and final steady states and hence the circulation at infinity remains unchanged. The effective duration of the transient state is inversely proportional to the square of the suction velocity.

The solutions obtained for the circular cylinder can be applied to the analogous problem of the non-steady flow past an infinite porous plate and for this case they are exact.

1. Introduction

In a previous paper (**1**), solutions were obtained for the steady circulatory flow about a circular cylinder, through the surface of which uniformly distributed suction was applied. In the general case, the cylinder was rotated at a steady, arbitrary, angular velocity and there was an arbitrary circulation at infinity, but the suction had to be greater than a certain limiting value for this circulation to exist.

A proposed method of generating this circulation was to rotate the cylinder with the suction off, thus setting up the well-known 'free vortex' flow, for which the circulation is constant at all radii. Suction, greater than the limiting value, was then applied and the speed of rotation of the cylinder was then altered to the chosen value.

Arguments were given to show that the circulation at infinity would remain unchanged throughout this process and indefinitely. These were based on considerations of the flux of vorticity by convection and diffusion across a fixed circuit and depended on the hypothesis that, during the change from one steady state to another, the vorticity was effectively zero outside a finite radius.

It is desirable to establish the constancy of the circulation at infinity more firmly, since this is of importance in connexion with the proposed

use of the Thwaites flap (**2**) as an anti-gust device. In reference **1** it was shown how the Thwaites flap could be used as an alternative method of generating the circulation in the problems discussed in that paper.

In this paper approximate solutions are obtained for the transient state occurring when the angular velocity of the cylinder and/or the rate of suction is *suddenly* changed. The suction is assumed to be always sufficiently large for the asymptotic exponential velocity distribution to exist in the initial and final states. These solutions are applicable to the infinite porous plate and are then exact.

Since the appearance of this paper in its original form as A.R.C. 12040, B. Thwaites (**3**) has dealt with the same problem and has obtained a solution in series, whereas the present results are given in a closed form.

2. Notation

r	radius
q	circumferential velocity component at radius r
a	radius of permeable circular cylinder
w_0	suction velocity normal to surface
ν	kinematic viscosity
R	$w_0 a/\nu$
t	time
q_a	peripheral velocity of cylinder
q_0	reference velocity $= K_\infty/(2\pi a)$ in case of circular cylinder
$\bar{q}$	q/q_0
T	$\nu t/a^2$
x	r/a
y	$x-1 = (r-a)/a$
C	q_0/q_a
K_∞	circulation at $r = \infty$
K_a	circulation at $r = a$
k	a constant
N	$(R-2)/2$
$\bar{q}_T$	the part of $\bar{q}$ dependent on the time
n	distance from the surface
y'	$q_0 n/\nu$
T'	tq_0^2/ν.

3. Theory

3.1. *The equation governing the motion*

As in reference **1**, we assume the motion to be independent of θ the angular coordinate and a function of the time t and the radius r.

The equation governing the circumferential velocity component q is (1)

$$\frac{1}{\nu}\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial r^2} + (R+1)\frac{1}{r}\frac{\partial q}{\partial r} + (R-1)\frac{q}{r^2}, \tag{1}$$

where

$$R = \frac{w_0 a}{\nu}, \tag{2}$$

a is the radius of the cylinder and w_0 is the suction velocity at the surface.

Introduce non-dimensional variables $\bar{q}$, x, T, such that

$$\left.\begin{aligned} q &= \bar{q}q_0 \\ r &= xa \\ t &= T\frac{a^2}{\nu} \end{aligned}\right\} \tag{3}$$

in which q_0 is a convenient reference velocity and is constant. Then (1) becomes

$$\frac{\partial \bar{q}}{\partial T} = \frac{\partial^2 \bar{q}}{\partial x^2} + \frac{(R+1)}{x}\frac{\partial \bar{q}}{\partial x} + \frac{(R-1)}{x^2}\bar{q}. \tag{4}$$

Assume a particular solution in the form

$$\bar{q} = e^{-k^2 T} x^{-R/2} f(x). \tag{5}$$

Substituting in (4) we obtain

$$x^2\frac{d^2 f}{dx^2} + x\frac{df}{dx} + (k^2x^2 - N^2)f = 0, \tag{6}$$

where

$$N = \frac{(R-2)}{2}, \tag{7}$$

and putting

$$x_1 = kx \tag{8}$$

(6) reduces to the standard form of Bessel's equation

$$x_1^2\frac{d^2 f}{dx_1^2} + x_1\frac{df}{dx_1} + (x_1^2 - N^2)f = 0. \tag{9}$$

Hence a solution of (4) is

$$\bar{q} = e^{-k^2 T} x^{-R/2}\{AJ_N(kx) + BY_N(kx)\}. \tag{10}$$

The steady state solution of reference **1** is also a solution of (4). This in the general case was expressed as

$$\frac{q}{q_a} = Cx^{-1} + (1-C)x^{-(R-1)}, \tag{11}$$

where

$$C = \frac{K_\infty}{K_a} = \frac{2\pi a q_0}{2\pi a q_a}, \tag{12}$$

q_a is the peripheral speed of the cylinder and q_0 is given by

$$q_0 = \frac{K_c}{2\pi a}. \tag{13}$$

This we choose as our reference velocity. Hence

$$\bar{q} = \frac{q}{q_0} = x^{-1} + \left(\frac{1}{C} - 1\right) x^{-(R-1)} \tag{14}$$

is a solution of (4).

If suffix (1) denotes the initial steady state at $T = 0$

$$\bar{q}_1 = x^{-1} + \left(\frac{1}{C_1} - 1\right) x^{-(R_1-1)} \tag{15}$$

and C, R refer to $0 < T$, then (14) will represent the final steady state.

Equations (14) and (10) must be combined in such a manner that

$$\left.\begin{aligned} q &= q_a, \quad x = 1 \quad \text{for all } T \\ \bar{q} &= \bar{q}_1, \quad T = 0 \end{aligned}\right\}. \tag{16}$$

The building up of a solution to satisfy the boundary conditions and the end conditions appears to be complicated and we therefore proceed to simplify (4) by assuming R large.

3.2. *Solution for R large*

If R is large, we see from the graphs of reference **1** or equation (14) for the steady state solution that the rapid changes of velocity occur near the surface, $x \to 1{\cdot}0$. Hence in (4) the term $(R-1)\bar{q}/x^2$ is negligible compared with the remaining terms and $(R+1)$ may be replaced by R. Also x, where it occurs explicitly, may be replaced by unity.

Put

$$x = 1 + y, \tag{17}$$

then (4) simplifies to

$$\frac{\partial \bar{q}}{\partial T} = \frac{\partial^2 \bar{q}}{\partial y^2} + R\frac{\partial \bar{q}}{\partial y}. \tag{18}$$

A particular solution of (18) is obtained as follows. Assume

$$\bar{q} = e^{-(k^2 T + \frac{1}{2}Ry)} f(y), \tag{19}$$

then (18) becomes

$$\frac{d^2 f}{dy^2} + \alpha^2 f = 0, \tag{20}$$

where

$$\alpha^2 = k^2 - \tfrac{1}{4}R^2. \tag{21}$$

Equation (20) occurs in diffusion problems and a simple solution is

$$f = A' \cos \alpha(y - \beta), \tag{22}$$

where A' and β are constants.

Putting

$$A' = \frac{F(\beta)}{\pi},$$

the solution of (18) is

$$\bar{q} = \frac{F(\beta)}{\pi} e^{-\{(\alpha^2+\frac{1}{4}R^2)T+\frac{1}{2}Ry\}} \cos\alpha(y-\beta). \tag{23}$$

Generalizing this, we may write

$$\bar{q} = \frac{1}{\pi}\exp\{-(\tfrac{1}{4}R^2T+\tfrac{1}{2}Ry)\} \int_0^\infty e^{-\alpha^2T}\, d\alpha \int_{-\infty}^{\infty} F(\beta)\cos\alpha(y-\beta)\, d\beta \tag{24}$$

as a particular solution of (18), where $F(\beta)$ is to be determined later from the end conditions via the Fourier integral theorem.

The steady state solution, corresponding to (15), which is a solution of (18) is

$$\bar{q} = 1+\left(\frac{1}{C}-1\right)e^{-Ry}, \tag{25}$$

and for the initial steady state solution, $T = 0$, we take

$$\bar{q}_1 = 1+\left(\frac{1}{C_1}-1\right)e^{-R_1y}. \tag{26}$$

If C, R refer to $0 < T$, then (25) is the final steady state, $T = \infty$. The change from C_1 to C represents a sudden change in peripheral speed (see (12)) and the change from R_1 to R represents a sudden change of suction velocity (see (2)). In the general case these can occur together.

We therefore take as the solution of (18) the sum of the solutions (24) and (25), i.e.

$$\bar{q} = \frac{1}{\pi}\exp\{-(\tfrac{1}{4}R^2T+\tfrac{1}{2}Ry)\} \int_0^\infty e^{-\alpha^2T}\, d\alpha \int_{-\infty}^{\infty} F(\beta)\cos\alpha(y-\beta)\, d\beta + \\ +1+\left(\frac{1}{C}-1\right)e^{-Ry}. \tag{27}$$

For $T = \infty$, this gives the final steady state (26). For $T = 0$, $\bar{q} = \bar{q}_1$. Hence from (26) and (27)

$$\left(\frac{1}{C_1}-1\right)e^{-R_1y}-\left(\frac{1}{C}-1\right)e^{-Ry} = \frac{1}{\pi}e^{-\frac{1}{2}Ry}\int_0^\infty d\alpha \int_{-\infty}^{\infty} F(\beta)\cos\alpha(y-\beta)\, d\beta.$$

or

$$\left(\frac{1}{C_1}-1\right)e^{-(R_1-\frac{1}{2}R)y}-\left(\frac{1}{C}-1\right)e^{-\frac{1}{2}Ry} = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} F(\beta)\cos\alpha(y-\beta)\, d\beta. \tag{28}$$

Hence by the Fourier integral theorem

$$F(\beta) = \left(\frac{1}{C_1}-1\right)e^{-(R_1-\frac{1}{2}R)\beta}-\left(\frac{1}{C}-1\right)e^{-\frac{1}{2}R\beta}, \tag{29}$$

where, so far, $F(\beta)$ is only defined for $0 < \beta < \infty$.

The boundary condition at the surface requires $q = q_a$, $y = 0$ for all $T > 0$, i.e. $\bar{q} = 1/C$. Hence from (27)

$$\int_{-\infty}^{\infty} F(\beta)\cos\alpha\beta \, d\beta = 0. \tag{30}$$

This is achieved, since $\cos\alpha\beta$ is even in β, by making $F(\beta)$ odd in β. Hence for

$$\left.\begin{aligned} -\infty < \beta < 0 \\ F(-\beta) = -F(\beta) \end{aligned}\right\}, \tag{31}$$

where $F(\beta)$ is given by (24) for $0 < \beta < \infty$.

Hence (27), together with (29) and (31), is the required solution of (18), satisfying the boundary conditions and the end conditions.

Writing

$$\bar{q}_T = \frac{1}{\pi}\exp\{-(\tfrac{1}{4}R^2T + \tfrac{1}{2}Ry)\}\int_0^{\infty} e^{-\alpha^2 T}\, d\alpha \int_{-\infty}^{\infty} F(\beta)\cos\alpha(y-\beta)\, d\beta \tag{32}$$

and assuming that the order of integration can be changed, we integrate with respect to α, using the result

$$\int_0^{\infty} e^{-z^2}\cos 2bz \, dz = \frac{\sqrt{\pi}}{2}e^{-b^2} \tag{33}$$

to obtain

$$\bar{q}_T = \frac{e^{-(\frac{1}{4}R^2T + \frac{1}{2}Ry)}}{2\sqrt{(\pi T)}}\int_{-\infty}^{\infty} F(\beta)e^{-(y-\beta)^2/4T}\, d\beta, \tag{34}$$

which, since $F(\beta)$ is odd in β, can be written

$$\begin{aligned}\bar{q}_T &= \frac{e^{-(\frac{1}{4}R^2T + \frac{1}{2}Ry)}}{2\sqrt{(\pi T)}}\int_0^{\infty} F(\beta)(e^{-(y-\beta)^2/4T} - e^{-(y+\beta)^2/4T})\, d\beta \\ &= \frac{e^{-\frac{1}{4}(RT^{\frac{1}{2}}+yT^{-\frac{1}{2}})^2}}{2\sqrt{(\pi T)}}\int_0^{\infty} e^{-\beta^2/4T}\left\{\left(\frac{1}{C_1}-1\right)e^{-(R_1-\frac{1}{2}R)\beta} - \right. \\ &\qquad \left. -\left(\frac{1}{C}-1\right)e^{-\frac{1}{2}R\beta}\right\}\{e^{(y/2T)\beta} - e^{-(y/2T)\beta}\}\, d\beta \end{aligned} \tag{35}$$

on substituting for $F(\beta)$ from (29).

Multiplying out the brackets and putting

$$\beta_1 = \frac{\beta}{2T^{\frac{1}{2}}} \tag{36}$$

(35) gives, on rearrangement,

$$\bar{q}_T = \frac{e^{-B^2}}{\sqrt{\pi}}\left\{\left(\frac{1}{C_1}-1\right)\left[e^{E^2}\int_0^\infty e^{-(\beta_1+E)^2}\,d\beta_1 - e^{D^2}\int_0^\infty e^{-(\beta_1+D)^2}\,d\beta_1\right] - \right.$$

$$\left. -\left(\frac{1}{C}-1\right)\left[e^{A^2}\int_0^\infty e^{-(\beta_1+A)^2}\,d\beta_1 - e^{B^2}\int_0^\infty e^{-(\beta_1+B)^2}\,d\beta_1\right]\right\}, \quad (37)$$

where

$$\left.\begin{aligned} A &= \tfrac{1}{2}(RT^{\frac{1}{2}} - yT^{-\frac{1}{2}}) &&= \tfrac{1}{2}(R^2T)^{-\frac{1}{2}}(R^2T - Ry)\\ B &= \tfrac{1}{2}(RT^{\frac{1}{2}} + yT^{-\frac{1}{2}}) &&= \tfrac{1}{2}(R^2T)^{-\frac{1}{2}}(R^2T + Ry)\\ D &= \tfrac{1}{2}\{(2R_1 - R)T^{\frac{1}{2}} + yT^{-\frac{1}{2}}\} &&= \tfrac{1}{2}(R^2T)^{-\frac{1}{2}}\left\{\left(\frac{2R_1}{R}-1\right)R^2T + Ry\right\}\\ E &= \tfrac{1}{2}\{(2R_1 - R)T^{\frac{1}{2}} - yT^{-\frac{1}{2}}\} &&= \tfrac{1}{2}(R^2T)^{-\frac{1}{2}}\left\{\left(\frac{2R_1}{R}-1\right)R^2T - Ry\right\} \end{aligned}\right\}. \quad (38)$$

The integrals in (37) can be expressed as error functions. Thus

$$\bar{q}_T = \frac{e^{-B^2}}{2}\left\{\left(\frac{1}{C_1}-1\right)[e^{E^2}(1-\operatorname{erf} E) - e^{D^2}(1-\operatorname{erf} D)] - \right.$$

$$\left. -\left(\frac{1}{C}-1\right)[e^{A^2}(1-\operatorname{erf} A) - e^{B^2}(1-\operatorname{erf} B)]\right\}, \quad (39)$$

and so from (27)
$$\bar{q} = \bar{q}_T + \left\{1+\left(\frac{1}{C}-1\right)e^{-Ry}\right\}. \quad (40)$$

Check. That this is a solution of (18) is easily checked, since each of the terms for $\bar{q}_T$ in (39) satisfies the equation.

Putting $y = 0$, makes $\bar{q}_T$ vanish and so from (40)

$$\bar{q} = \frac{1}{C}, \quad \text{i.e.} \quad q = q_a \quad \text{as required.}$$

Putting $y = \infty$ makes $\bar{q}_T$ vanish and again from (40)

$$\bar{q} = 1, \quad \text{i.e.} \quad q = q_0 \quad \text{as required.}$$

Putting $T = \infty$ we are left with the final steady state solution

$$q = 1 + \left(\frac{1}{C}-1\right)e^{-Ry}.$$

Putting $T = 0$, after multiplying through by e^{-B^2}, in (39), gives

$$(\bar{q}_T)_{T=0} = \frac{1}{2}\left\{\left(\frac{1}{C_1}-1\right)2e^{-R_1 y} - \left(\frac{1}{C}-1\right)2e^{-Ry}\right\}$$

and substituting in (40) we obtain

$$\bar{q} = 1 + \left(\frac{1}{C_1}-1\right)e^{-R_1 y} = \bar{q}_1$$

as required.

The velocity distribution given by (38), (39), and (40) is seen to be a function of C_1, C, R_1/R, R^2T, and Ry. The C_1 and C refer to the initial and final peripheral velocities of the boundary; R_1/R relates the initial and final suction velocities, for, from (2),

$$\frac{R_1}{R} = \frac{(w_0)_1 a}{\nu} \frac{\nu}{w_0 a} = \frac{(w_0)_1}{w_0}. \tag{41}$$

R^2T is, from (2) and (3), given by

$$R^2T = \frac{w_0^2 a^2}{\nu^2} \frac{\nu t}{a^2} = \frac{w_0^2 t}{\nu}. \tag{42}$$

If n denotes the distance measured from the surface, then

$$y = \frac{r-a}{a} = \frac{n}{a}, \tag{43}$$

so that

$$Ry = \frac{w_0 a}{\nu} \frac{n}{a} = \frac{w_0 n}{\nu}. \tag{44}$$

Hence, we see that R enters the problem only in ensuring that the exponential velocity distribution is attained. When this is so, the radius a of the cylinder is of no significance and the essential non-dimensional parameters, describing the velocity distribution $\bar{q}$, are

$$C_1, \quad C, \quad \frac{(w_0)_1}{w_0}, \quad \frac{w_0^2 t}{\nu} \quad \text{and} \quad \frac{w_0 n}{\nu}.$$

3.3. *Particular cases*

(*a*) $\begin{cases} R_1 = R\text{: suction kept constant.} \\ C_1 = 1\text{: peripheral speed to generate appropriate circulation, i.e.} \\ \qquad q_a = q_0 \text{ at start, } T = 0. \\ C = \infty\text{: peripheral speed zero, i.e. } q_a = 0,\ T > 0. \end{cases}$

Then

$$\left.\begin{aligned} A = E &= \frac{(R^2T)^{-\frac{1}{2}}}{2}(R^2T - Ry) \\ B = D &= \frac{(R^2T)}{2}(R^2T + Ry), \end{aligned}\right\} \tag{45}$$

and from (39) and (40)

$$\bar{q} = \tfrac{1}{2}\{e^{-Ry}(1-\operatorname{erf} A) - (1-\operatorname{erf} B)\} + (1-e^{-Ry}). \tag{46}$$

(*b*) $\begin{cases} R_1 \neq R\text{: sudden change of suction flow at } T = 0. \\ C_1 = C = \infty\text{: cylinder at rest at all } T. \end{cases}$

Then A, B, D, and E remain as given by (38), and from (39) and (40)

$$\bar{q} = \frac{e^{-B^2}}{2}\{e^{A^2}(1-\operatorname{erf} A) - e^{B^2}(1-\operatorname{erf} B) + e^{D^2}(1-\operatorname{erf} D) - e^{E^2}(1-\operatorname{erf} E)\} + \\ + (1-e^{-Ry}). \tag{47}$$

If $R_1 = \infty$: infinite suction, then (47) reduces to (46) with A and B given by (45).

4. The infinite flat plate with uniform suction

Keeping to the present notation, let w_0 be the suction velocity normal to the plate, q be the velocity parallel to the plate at a distance n from the surface, and q_0 the value of q for $n \to \infty$. Here q is assumed to be a function of n and t only. Then the equation of motion is

$$\frac{\partial q}{\partial t} = \nu \frac{\partial^2 q}{\partial n^2} + w_0 \frac{\partial q}{\partial n}. \tag{48}$$

Since there is no representative length associated with the plate, we introduce non-dimensional variables defined by

$$\left.\begin{aligned} q &= \bar{q} q_0 \\ n &= y' \frac{\nu}{q_0} \\ t &= T' \frac{\nu}{q_0^2} \end{aligned}\right\} \tag{49}$$

and (48) becomes

$$\frac{\partial \bar{q}}{\partial T'} = \frac{\partial^2 \bar{q}}{\partial y'^2} + \left(\frac{w_0}{q_0}\right) \frac{\partial \bar{q}}{\partial y'}. \tag{50}$$

This is of the same form as (18), the boundary conditions are the same, and so the solution—equations (38), (39), and (40)—will hold if we replace R by (w_0/q_0), T by T', and y by y'. The non-dimensional parameters given in equations (41), (42), and (44) still stand.

Since equation (48) is exact and no approximations have been made, the solution found for the cylinder and applied to the plate is exact. This arises from the fact that the exact steady state solutions take the exponential form in the case of the infinite plate.

5. Example

As a simple illustration of the change of velocity distribution with time, we take $\bar{q}$ to be given by (46), section 3.3. This velocity distribution arises in two different problems. In the first, the suction is kept constant, but the angular velocity of the cylinder is suddenly reduced to zero from the value necessary to generate the circulation at infinity. In the second, the cylinder is at rest with a finite circulation at infinity and the suction is suddenly reduced from an infinite value to a finite value. Corresponding flows exist for the infinite flat plate; $\bar{q} = q/q_0$ is then a function of $w_0^2 t/\nu$ and $w_0 n/\nu$. In Fig. 1 $\bar{q}$ is plotted as a function of $w_0 n/\nu$ for a few chosen values of $w_0^2 t/\nu$. When $w_0^2 t/\nu \geqslant 9{\cdot}0$, $\bar{q}$ is everywhere within 0·002 of the steady state value $t \to \infty$. It will be noted that the velocity distributions lie between the initial and final velocity distributions.

In the case of the cylinder $R = w_0 a/\nu > 100$ for the exponential form to be attained (see reference **1**). To take a numerical example, with $w_0^2 t/\nu = 9{\cdot}0$, if $a = \frac{1}{8}$ ft., $\nu = 1{\cdot}59 \times 10^{-4}$ ft.^{2}/sec., then $w_0 \geqslant \frac{1}{8}$ ft./sec. for $R \geqslant 100$. For $w_0 = \frac{1}{8}$ ft./sec. $t = 9{\cdot}0\nu/w_0^2 \doteqdot 0{\cdot}09$ sec.

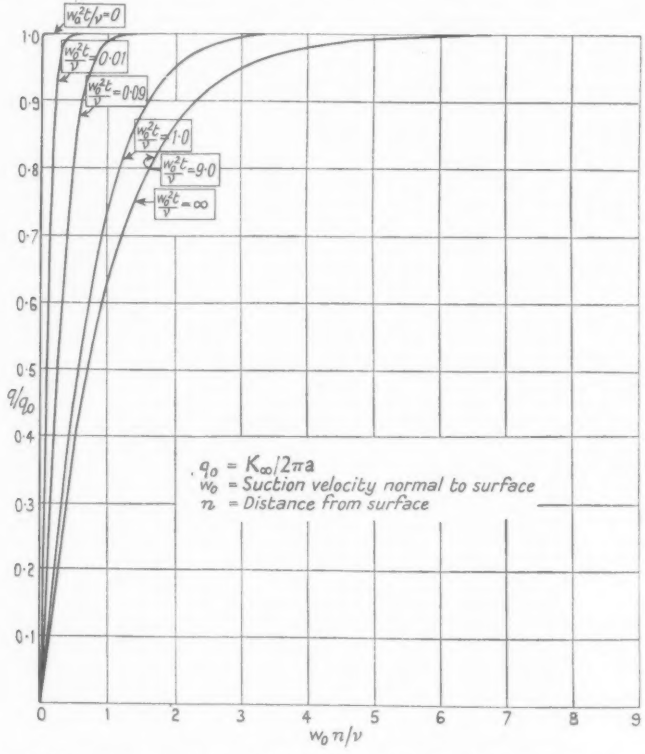

FIG. 1. Velocity distribution q/q_0 v. $w_0 n/\nu$ at various times t, (46), for (*a*) rotating cylinder with steady circulatory flow: suction applied, then cylinder suddenly brought to rest; or (*b*) cylinder at rest with steady circulatory flow and infinite suction, then suction suddenly reduced to finite value.

In the case of the infinite plate, if we take a free stream velocity $q_0 = 200$ ft./sec., then the times for the final steady state to be effectively reached for various values of w_0/q_0 are as follows:

$\frac{w_0}{q_0} = 10^{-3}$	w_0(ft./sec.) $= 0{\cdot}2$	t(sec.) $\doteqdot 0{\cdot}036$
$= 10^{-4}$	$= 0{\cdot}02$	$\doteqdot 3{\cdot}6$
$= 10^{-5}$	$= 0{\cdot}002$	$\doteqdot 360.$

6. Conclusions

Provided the suction is sufficiently large, solutions can be obtained for the transient state occurring when the angular velocity of the circular cylinder and/or the rate of suction is suddenly changed.

The velocity distributions lie between those of the initial and final steady state solutions and the circulation at infinity is unchanged. It seems likely that this will be so for smaller suctions, provided $R\ (= w_0 a/\nu) > 2$.

The proposed method of generating the steady flows described in reference **1** should therefore work, and it is highly probable that they could be generated by means of the Thwaites flap as also described in reference **1**. If this is so, then in the case of a wing with rounded trailing edge, the Thwaites flap, having generated a circulation, can be withdrawn into the wing and the circulation will remain independent of velocity and incidence changes. The lift will depend only on the forward speed and will scarcely be affected by gusts, as was pointed out by Thwaites (**2**).

The solutions obtained for the circular cylinder with R large are applicable to the infinite flat plate for all suction velocities and are exact.

The effective duration of the transient state is inversely proportional to the square of the suction velocity and for normal rates of suction is quite short.

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FLUID MOTIONS WHOSE KINEMATICS ARE INDEPENDENT OF THE COMPRESSIBILITY OF THE FLUID

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SUMMARY

The following question is studied: What steady motions of a compressible fluid may be reproduced exactly by the motion of an incompressible fluid? More precisely, what vector functions of position exist, representing simultaneously the velocity in a possible motion of a compressible fluid, and the velocity in a possible motion of an incompressible fluid? A general account is given of the theory of motions of this type. Two-dimensional motion is then studied in detail, and it is shown that in order that a two-dimensional fluid motion may be of the required type, it is necessary and sufficient that the streamlines be either concentric circles or parallel straight lines. A very special case of three-dimensional motion, in which the streamlines are parallel straight lines, is then mentioned. Finally a general discussion is given of the type of results obtained, and it is shown how the restrictions encountered are interpreted by means of the theory of partial differential equations.

1. General theory

We consider the steady motions of two non-viscous fluids; firstly a compressible fluid, whose pressure is related to its density by the adiabatic or other prescribed law; secondly a uniform incompressible fluid. In either case let p, ρ, $\mathbf{u}$ denote respectively the pressure, density, and velocity vector at a point whose position vector is $\mathbf{r}$. Then the object of this investigation is to determine in what types of motion, if any, the same vector function of position $\mathbf{u}(\mathbf{r})$ can represent the velocity in both cases. Thus the problem is to find a single vector function $\mathbf{u}(\mathbf{r})$, such that the equation of continuity and the dynamical equations of motion for either fluid may be satisfied by a suitable choice of the (scalar) pressure function $p(\mathbf{r})$ in each case. For the sake of brevity we will henceforth refer to the compressible fluid problem as case (a), and to the incompressible fluid problem as case (b).

In either case the equation of continuity is

$$\operatorname{div}(\rho\mathbf{u}) = 0,$$

which in case (a) may be written

$$\rho \operatorname{div} \mathbf{u} + \mathbf{u}.\operatorname{grad}\rho = 0, \tag{1}$$

and in case (b) becomes

$$\operatorname{div}\mathbf{u} = 0. \tag{2}$$

Since $\mathbf{u}$ must satisfy both (1) and (2), we have in case (a)

$$\mathbf{u}.\operatorname{grad}\rho = 0, \tag{3}$$

showing that the density is constant along any streamline. Now the dynamical equations of motion, in either case, may be written vectorially as

$$\operatorname{grad}\left\{\tfrac{1}{2}\mathbf{u}^2 + \int (1/\rho)\,dp\right\} = \mathbf{u} \wedge \operatorname{curl}\mathbf{u}. \tag{4}$$

Forming the scalar product of this equation with $\mathbf{u}$ we obtain in both cases

$$\mathbf{u}.\operatorname{grad}(\tfrac{1}{2}\mathbf{u}^2) + (1/\rho)\mathbf{u}.\operatorname{grad}p = 0, \tag{5}$$

which in case (a) may be written

$$\mathbf{u}.\operatorname{grad}(\tfrac{1}{2}\mathbf{u}^2) + (1/\rho)(dp/d\rho)\mathbf{u}.\operatorname{grad}\rho = 0. \tag{6}$$

Thus from (3) and (6) we see that

$$\mathbf{u}.\operatorname{grad}(\tfrac{1}{2}\mathbf{u}^2) = 0, \tag{7}$$

showing that the velocity is constant in magnitude along any streamline.

Next, taking the curl of (4), we obtain in both cases the well-known necessary condition for a steady fluid motion,

$$\operatorname{curl}\{\mathbf{u} \wedge \operatorname{curl}\mathbf{u}\} = 0, \tag{8}$$

and we will now see that the conditions (2), (7), and (8) are sufficient for the solution of the problem. For suppose that $\mathbf{u}(\mathbf{r})$ has been found, satisfying these three equations. Then (8) is the sufficient condition for the existence of a scalar function of position, $\chi(\mathbf{r})$, such that

$$\mathbf{u} \wedge \operatorname{curl}\mathbf{u} = \operatorname{grad}\chi;$$

and hence we may solve (4) for $\int (1/\rho)\,dp$ in terms of the position vector $\mathbf{r}$. Thus

$$\int (1/\rho)\,dp = \chi - \tfrac{1}{2}\mathbf{u}^2 + \text{constant}. \tag{9}$$

In case (a), $\int (1/\rho)\,dp$ is a known function of ρ, whose derivative $(1/\rho)\,dp/d\rho$ is not zero, and hence (9) gives ρ, and therefore p, in terms of $\mathbf{r}$. While in case (b), $\int (1/\rho)\,dp = p/\rho_0 + \text{constant}$, ρ_0 being the fixed density, so that once again from (9) we may find p in terms of $\mathbf{r}$. In each case, (4) being now satisfied by the appropriate choice of the pressure function, we obtain (5) as before, and hence in case (a), (6). From (6) and (7), which is by hypothesis satisfied, we deduce (3), in case (a); and (2) being also satisfied, (1) then follows. Thus all the conditions in both cases are fulfilled.

There are several applications of the foregoing theory in which large simplifications of the conditions obtained exist. In motion in three dimensions, with symmetry about an axis, (2) is automatically satisfied by the use of Stokes' stream function Ψ. If r be the radial distance from the axis, and ζ the vorticity, (8) gives the well-known result that ζ/r is constant along any streamline; and hence that ζ/r is a function of Ψ. The condition

(7) then expresses the fact that the magnitude of the velocity is a function of Ψ; from these last two conditions, therefore, two partial differential equations, one of the second order and one of the first order, are obtained for the determination of Ψ.

In two-dimensional motion the simplifications are much greater; and this problem, and also a special case of motion in three dimensions, will now be treated in detail.

2. Two-dimensional motion

Let u, v be the components of velocity parallel to the rectangular axes of x, y respectively, and let ζ be the vorticity, so that $2\zeta = (\partial v/\partial x) - (\partial u/\partial y)$. Then (2) is satisfied by the use of the stream function ψ, so that

$$u = -\partial\psi/\partial y; \qquad v = \partial\psi/\partial x; \qquad 2\zeta = \nabla^2\psi;$$

while (7), which exhibits the constancy of the velocity along a streamline, shows that the velocity must be a function of ψ. Thus

$$(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2 = \text{function of } \psi. \tag{10}$$

In this case (8) becomes a single scalar equation,

$$u\,\partial\zeta/\partial x + v\,\partial\zeta/\partial y + \zeta\{\partial u/\partial x + \partial v/\partial y\} = 0,$$

or using (2)

$$u\,\partial\zeta/\partial x + v\,\partial\zeta/\partial y = 0, \tag{11}$$

which expresses the standard condition that the vorticity is constant along any streamline.

Now (10) is the well-known necessary and sufficient condition that the curves $\psi(x, y) = \text{constant}$, in this case the streamlines, shall be a family of 'parallel' curves; that is, curves having a common system of normals, and such that the distance along a common normal between any two chosen curves of the parallel system is constant. Let us then suppose that a family of parallel curves has been chosen. We now investigate the vorticity of motions having this set of curves as streamlines. Let Γ denote a certain streamline; and let θ be the angle between the x-axis and a tangent to Γ: and let Y be the (constant) normal distance between Γ and a selected reference curve of the parallel system. Then the equation to the tangent is

$$x\sin\theta - y\cos\theta - F(\theta) - Y = 0, \tag{12}$$

in which $F(\theta)$ is a function depending on the form of the parallel curves. Then except when θ is fixed, in which case the streamlines are parallel straight lines, Γ is the envelope of (12) as θ varies, and is thus given by (12) and

$$x\cos\theta + y\sin\theta - F'(\theta) = 0; \tag{13}$$

which is clearly the equation to one of the common normals to the parallel curves. The coordinates of the point of contact of the tangent to Γ, i.e. the

intersection of the tangent and normal, are given by solving (12) and (13) for x and y. Alternatively we may regard (12) and (13) as expressing Y and θ in terms of x and y; hence differentiating and using the equations themselves we obtain

$$\left.\begin{aligned}(\partial Y/\partial x)^2+(\partial Y/\partial y)^2 &= 1,\\ (\partial^2 Y/\partial x^2)+(\partial^2 Y/\partial y^2) &= 1/\{Y+F(\theta)+F''(\theta)\}.\end{aligned}\right\} \tag{14}$$

But since the streamlines are given by $\psi(x,y) =$ constant, there must be a unique correspondence between ψ and Y; so that ψ must be a function of Y, such that $d\psi/dY \neq 0$. Thus

$$2\zeta = \nabla^2\psi = (d\psi/dY)\nabla^2 Y+(d^2\psi/dY^2)\{(\partial Y/\partial x)^2+(\partial Y/\partial y)^2\}$$

or by (14), $$2\zeta = (d^2\psi/dY^2)+(d\psi/dY)/\{Y+F(\theta)+F''(\theta)\}. \tag{15}$$

Therefore in order that the condition of the constancy of ζ along a streamline may be satisfied, that is, in order that (15) may be independent of θ, it is necessary and sufficient that

$$F''(\theta)+F(\theta) = c, \tag{16}$$

where c is a constant. The general integral of (16) is

$$\left.\begin{aligned}F(\theta) &= g\sin\theta-f\cos\theta+c,\\ F'(\theta) &= g\cos\theta+f\sin\theta,\end{aligned}\right\} \tag{17}$$

where g and f are also constants. Then substituting from (17) in (12) and (13) we have for the tangent and normal respectively

$$(x-g)\sin\theta-(y-f)\cos\theta = Y+c, \tag{12'}$$

$$(x-g)\cos\theta+(y-f)\sin\theta = 0. \tag{13'}$$

Thus squaring and adding we obtain

$$(x-g)^2+(y-f)^2 = (Y+c)^2, \tag{18}$$

showing that the streamlines are concentric circles, the centre being the arbitrary point (g,f).

In the special case when θ is fixed and the streamlines are parallel straight lines, given by (12), we obtain directly from (12)

$$(\partial Y/\partial x)^2+(\partial Y/\partial y)^2 = 1; \qquad (\partial^2 Y/\partial x^2)+(\partial^2 Y/\partial y^2) = 0,$$

and thus $$2\zeta = d^2\psi/dY^2, \tag{15'}$$

showing that ζ is indeed constant along a streamline.

Thus we see that the necessary and sufficient condition for a two-dimensional motion of the type required is that the streamlines shall be either concentric circles or parallel straight lines. Important special cases are the irrotational motion generated by a line-vortex, and the motion produced by a vortex of circular section.

3. A special three-dimensional motion

We now examine the situation which arises in three dimensions when the velocity vector is fixed in direction, which, of course, implies that the streamlines are parallel straight lines. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be three orthogonal unit vectors, parallel respectively to the rectangular axes of x, y, z; and let the velocity be always parallel to the axis of x, so that

$$\mathbf{u} = u(x, y, z)\mathbf{i}.$$

Then (2) gives $\partial u/\partial x = 0$, showing that the velocity is a function of y and z only; and (7) is then also satisfied. For the remaining condition we have

$$\begin{aligned} \operatorname{curl} \mathbf{u} &= (\partial u/\partial z)\mathbf{j} - (\partial u/\partial y)\mathbf{k}, \\ \mathbf{u} \wedge \operatorname{curl} \mathbf{u} &= (u\, \partial u/\partial y)\mathbf{j} + (u\, \partial u/\partial z)\mathbf{k} \\ &= \operatorname{grad}(\tfrac{1}{2}\mathbf{u}^2), \end{aligned}$$

and thus (8) is always satisfied. Thus, for example, any steady flow of a gas in a straight pipe, in which the velocity varies over the cross-section in any manner, may be kinematically reproduced by the flow of a liquid in the pipe.

4. General conclusions

It seems at first sight rather surprising that the results of the investigation into the two-dimensional case should be so restricted. We have in fact seen that the possible streamline systems are obtained by equating to a constant the simple function on the left-hand side of (18), which contains only two arbitrary constants. Now the condition (8) applies to all steady fluid motions and (2) applies to all motions of an incompressible fluid. Furthermore, although we have added the condition (7), the motion has not been restricted to being irrotational. Thus the results might have been expected to be of some generality.

The explanation lies in the fact that while the condition for irrotational motion, $\operatorname{curl} \mathbf{u} = 0$, involves (8) as a direct consequence, the condition (7) does not. If, with the notation of section 2, we substitute for u and v in terms of ψ in (7), we obtain a partial differential equation of the second order for ψ, of which (10) is an intermediate integral. Again, if we substitute for u, v, and ζ in terms of ψ in (11), which is equivalent to (8), we obtain a partial differential equation of the third order, of which the condition of the constancy of ζ along a streamline is an intermediate integral. Thus fundamentally the two-dimensional problem is that of finding the common solutions to two partial differential equations, one of the second and one of the third order; and it is a well-known result in the theory of differential equations with two independent variables, that

except when special conditions are satisfied, the solutions to this problem depend upon not more than six arbitrary constants: this is entirely in accordance with the result obtained directly in section 2.

Similar remarks apply to the case of motion with axial symmetry. It may readily be seen that the conditions mentioned briefly at the end of section 1 are intermediate integrals of the equations which express the fundamental conditions. These are once again of the second and third orders, so that just as in the two-dimensional case we should expect the possible solutions to be of an extremely simple and restricted nature.

THE CHARACTERISTICS OF SYSTEMS WHICH ARE NEARLY IN A STATE OF NEUTRAL STATIC STABILITY

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SUMMARY

It is shown that the rate of subsidence or divergence λ of a system which is near a state of neutral static stability can easily be calculated from a knowledge of the mode of displacement in the neutral state and this mode is found by solving a set of linear algebraic equations. The first-order correction to the mode can also be found and in important cases this can be made the basis for calculating a second approximation to λ; if necessary a further correction to the mode can now be found and from this a still more accurate root can be calculated. The method can be extended to continuous systems having infinitely many degrees of freedom.

1. Introduction

SYSTEMS which are nearly in a state of neutral static stability are of considerable interest and technical importance. Exactly neutral systems are specially simple inasmuch as the neutral mode of displacement can be found merely by solving a set of simultaneous linear algebraic equations. It is shown in this paper that the small rate of divergence or subsidence λ of a nearly neutral system can be very easily calculated to the first order of small quantities from a knowledge of the neutral mode; the first-order corrections to this mode can readily be found also. When the matrices of the inertias, dampings, and stiffnesses are symmetric the mode correct to the first order can be made the basis of a calculation of λ correct to the second order. For a continuous system with infinitely many freedoms the first-order approximation to λ can be derived from the values of the potential energy and rate of dissipation of energy in an imaginary divergent motion proportional to the neutral mode and having λ equal to unity.

2. Exactly neutral systems

The matrices of the inertias, damping coefficients, and stiffnesses are, as usual, denoted by A, B, and C respectively. These are square matrices of order n, where n is the number of degrees of freedom of the system. The static displacements q corresponding to static loads Q are given by

$$Cq = Q \tag{2.1}$$

or

$$q = C^{-1}Q. \tag{2.2}$$

If the determinant of C becomes very small, the elements of C^{-1} become

very large and the displacements q corresponding to given loads Q also become very large. When $|C|$ vanishes the system is in a state of neutral static stability and then finite displacements can occur in the absence of applied loads.* Since the constant term in the determinantal equation of the system is equal to $|C|$, the equation has the root $\lambda = 0$ when $|C|$ vanishes. Thus the velocities $\dot{q}$ and accelerations $\ddot{q}$ are zero in the mode of displacement corresponding to the root $\lambda = 0$. There are, however, $(n-1)$ other modes which, in general, correspond to non-vanishing roots λ, and in these modes the velocities and accelerations are not, in general, zero. Cases of multiple zero roots can arise but are rare. When a displacement occurs in the neutral mode the forces are always balanced. Hence the total work done in a neutral displacement is zero. As an example, when the angle of incidence of an aircraft varies slowly when the centre of mass is at the longitudinal neutral point the resultant aerodynamic force is constantly equal to the weight of the aircraft.

3. Nearly neutral systems

Suppose next that the system is near but not in the state of neutral static stability. There are then some simple methods for finding the characteristics of the nearly neutral mode.

(*a*) *The matrices A, B, and C are all symmetric*

In this instance we can apply an extension of Rayleigh's Principle. Suppose that in a datum state of the system there is a characteristic root λ and a corresponding modal column q. Then λ must satisfy exactly the quadratic equation

$$\lambda^2(q'Aq)+\lambda(q'Bq)+q'Cq = 0, \qquad (3.1)$$

where the coefficients are quadratic forms in the elements of q. This equation is obtained from the dynamical equation in matrix form

$$\lambda^2 Aq+\lambda Bq+Cq = 0$$

by premultiplication by q', the transposed of q, i.e. the row matrix having the same elements as the column q; the equation is true whether A, B, and C are symmetric or not. Now let A, B, and C be symmetric, and let the elements of q depart from their true values by small quantities of the first order. Then according to the principle the value of the root λ obtained from the quadratic equation (3.1) will only be in error by a small quantity of the second order (**1**). Accordingly, let A, B, and C refer to the near neutral state and substitute for q the neutral modal column q_0 which is easily calculated. The root will be a small quantity of the first order and λ^2 will be of the same order as the terms neglected and therefore can be

* We can suppose this motion started by an extremely small impulse.

discarded. Hence we derive the approximation

$$\lambda = -\frac{q_0' C q_0}{q_0' B q_0} \tag{3.2}$$

which is easily computed.

Let C reduce to C_0 in the exactly neutral state so that

$$C_0 q_0 = 0, \tag{3.3}$$

and let also

$$C = C_0 + \delta C, \tag{3.4}$$

where the elements of δC are small quantities of the first order and C_0 is symmetric. Then by (3.3) and (3.4)

$$q_0' C q_0 = q_0' \delta C q_0 \tag{3.5}$$

and (3.2) becomes

$$\lambda = -\frac{q_0' \delta C q_0}{q_0' B q_0}, \tag{3.6}$$

which is correct to the first order of small quantities. From knowledge of this approximation to λ we can calculate the first-order correction to the modal column q_0 merely by solving a set of simultaneous linear equations (see equation (3.13)).

The approximation to λ can now be improved by use of the corrected modal column and the equation (3.1). Since λ is a small quantity of the first order, equation (3.1) will be correct to the second order if $q'Bq$ is correct to the first order and $q'Cq$ to the second order. Now suppose that all the elements of δC are proportional to p, which is thus of the first order, and let

$$\delta C = pC_1. \tag{3.7}$$

Suppose also that to the second order

$$q = q_0 + pq_1 + p^2 q_2, \tag{3.8}$$

where q_1 is now known but q_2 is not. Then to the second order of small quantities

$$q'Cq = p(q_0' C_0 q_1 + q_0' C_1 q_0) + p^2(q_0' C_0 q_2 + q_0' C_1 q_1 + q_1' C_0 q_1 + q_1' C_1 q_0) \tag{3.9}$$

in view of (3.3). But equation (3.3) gives by transposition

$$q_0' C_0 = 0 \tag{3.10}$$

since C_0 is symmetric. Hence (3.9) becomes

$$q'Cq = pq_0' C_1 q_0 + p^2(q_0' C_1 q_1 + q_1' C_0 q_1 + q_1' C_1 q_0). \tag{3.11}$$

Now this does not contain q_2, and we conclude that equation (3.1) will be correct to the second order of small quantities when we substitute for q the approximate modal column which is correct to the first order. Hence the small root of (3.1) will now be correct to the second order.

The improved approximation to λ obtained from the quadratic equation (3.1) can be used to obtain an improved approximation to the corresponding mode by use of the dynamical equations. Then this recalculated mode

may, if desired, be used to obtain a still better approximation to λ by means of (3.1). The whole process can be repeated as often as desired.

(*b*) *The matrices A, B, and C are general, except that C is nearly singular*

The dynamical equations in matrix form yield

$$(\lambda^2 A+\lambda B+C_0+\delta C)(q_0+\delta q) = 0, \tag{3.12}$$

while (3.3) is again satisfied. If we now neglect second-order terms we obtain

$$\lambda B q_0+C_0\,\delta q = -\delta C\, q_0. \tag{3.13}$$

This represents n linear scalar equations and there are apparently $(n+1)$ unknowns, namely λ and the n elements of δq. However, the modal column is arbitrary to a scalar multiplier and we may accordingly assign one of the elements of δq, say make one element zero. We have now n unknowns, which can be calculated.

We shall now show that when C_0 is symmetric this procedure leads exactly to the value of λ given by (3.2) or (3.6). Premultiply (3.13) by q_0'. We obtain

$$\begin{aligned}\lambda q_0'\, B q_0 &= -q_0'(C_0\,\delta q+\delta C\, q_0)\\ &= -q_0'\,\delta C\, q_0\end{aligned}$$

by (3.10) and this is identical with (3.6).

4. A simple numerical example

Let the inertia matrix be

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix}, \tag{4.1}$$

the damping matrix

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \tag{4.2}$$

and the stiffness matrix be

$$C = C_0+\delta C, \tag{4.3}$$

where

$$C_0 = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{4.4}$$

and is singular, while

$$\delta C = p \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.5}$$

where p is supposed to be small. The matrices are all symmetric so both methods described above are applicable. To facilitate comparisons with

the approximations which we shall obtain we give the expansion of the exact determinantal equation for λ, namely

$$|\lambda^2 A+\lambda B+C| \equiv 83\lambda^6+96\lambda^5+(16p+144)\lambda^4+ \\ +(14p+86)\lambda^3+(21p+33)\lambda^2+(8p+6)\lambda+p = 0. \quad (4.6)$$

When p is zero this has a zero root in accordance with C_0 being singular. The modal column q_0 corresponding to the zero root is obtained at once from the linear equation

$$C_0 q_0 = 0$$

and is found to be

$$q_0 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad (4.7)$$

where the last element has been assigned the value unity. When p is given the value 0·1 we find from (4.6) the true value of the numerically small root and the corresponding modal column to be

$$\lambda = -0{\cdot}01597_1 \quad (4.8)$$

$$q = \begin{bmatrix} -2{\cdot}22535 \\ 1{\cdot}13840 \\ 1{\cdot}0 \end{bmatrix} \quad (4.9)$$

to five places of decimals.

The first approximation to the small root as given by (3.6) is

$$\lambda = -\tfrac{1}{6}p = -0{\cdot}01667. \quad (4.10)$$

Equation (3.13) yields the same value of λ and when we assign the zero value to the final element in δq we obtain the approximate modal column

$$\begin{bmatrix} -2{\cdot}21667 \\ 1{\cdot}13333 \\ 1{\cdot}0 \end{bmatrix}. \quad (4.11)$$

On substitution of this value of q in equation (3.1) we obtain the quadratic equation

$$22{\cdot}12778\lambda^2+7{\cdot}20500\lambda+0{\cdot}10944_4 = 0, \quad (4.12)$$

of which the smaller root is

$$\lambda = -0{\cdot}01597_4. \quad (4.13)$$

This differs from the true root only by about 1 part in 5,000. The error in the modal column is of the order of $\frac{1}{2}$ per cent. and the departure from the neutral modal column is about 10 per cent.

For comparison, the calculation has been repeated with the inertia matrix reduced to one-quarter of that given in (4.1), i.e. with all the inertia coefficients reduced to one-quarter of their former values. The first-order approximations to λ and to the mode remain as in (4.10) and

(4.11) respectively. However, the quadratic equation (3.1) giving the second approximation to λ is obtained from (4.12) by reducing the coefficient of λ^2 to one-quarter of the value in (4.12) and the second approximation is

$$\lambda = -0{\cdot}01537_1. \tag{4.14}$$

Now the 'exact' root as obtained from the determinantal equation is

$$\lambda = -0{\cdot}01537_0, \tag{4.15}$$

so the error is only about 1 part in 15,000, or say one-third of the former error. The 'exact' mode is

$$q = \begin{bmatrix} -2{\cdot}22377 \\ 1{\cdot}13745 \\ 1{\cdot}0 \end{bmatrix}, \tag{4.16}$$

so the first approximation to the mode is in error by roughly 0·3 per cent.

5. Details of the procedure in dealing with a nearly neutral system

We suppose that the determinant $|C|$ of the stiffness matrix has been evaluated and found to be small, so the system is near a state of neutral static stability. It is now convenient to consider an artificial state of the system which is exactly neutral and we can derive this merely by varying *one* element of C; this element may be chosen so as to simplify the calculation. The increment of the chosen element is $-|C|/K$, where K is its cofactor. We next derive, merely by solving a set of linear equations, the neutral mode q_0 and may then proceed as explained in § 3. There is some advantage in choosing the element of the modal column which is arbitrarily kept fixed so that the product of this element and the varied element of C occurs in the calculations.

6. Treatment of continuous systems

The methods given above are not, in general, immediately applicable to continuous systems having infinitely many degrees of freedom. We may, however, treat such systems with any desired degree of approximation by the 'Lagrangian' method in which the continuous system is represented by another having a finite number of degrees of freedom (**2**). This method, also known as 'the method of semi-rigid representation', has been much used in dealing with aero-elastic problems and is of great utility. The methods explained in the present paper can be applied at once to the semi-rigid representation of the system.

Another method can be applied directly when the forces dependent on the displacements are derivable from a single-valued potential, for then the stiffness matrix is symmetric. We saw in section 3 (*b*), that equation (3.2) is valid to the first order of small quantities when C_0 is symmetric

whatever may be the nature of A and B. Now C_0 will be symmetric in the circumstances we have postulated so (3.2) is applicable however many degrees of freedom the system may possess. To assist the application of the formula we now give a physical interpretation of the numerator and denominator. We know that the potential energy stored in the system when the displacement is q_0 is

$$V = \tfrac{1}{2} q_0' C q_0. \tag{6.1}$$

Next, the time rate of dissipation of energy in the system is

$$D = \dot{q}' B \dot{q}, \tag{6.2}$$

and if we suppose that

$$q = q_0 e^{\lambda t} \tag{6.3}$$

this becomes

$$D(\lambda) = \lambda^2 q_0' B q_0. \tag{6.4}$$

Thus we see that

$$q_0' B q_0 = D(1), \tag{6.5}$$

where $D(1)$ is the rate of dissipation of energy when

$$q = q_0 e^{t}. \tag{6.6}$$

Accordingly equation (3.2) becomes

$$\lambda = -\frac{2V}{D(1)}. \tag{6.7}$$

The numerator and denominator can be calculated for a continuous system when the neutral mode of displacement is known.

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ON THE STRESSES IN CLOSE-COILED HELICAL SPRINGS

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SUMMARY

The problem of the stresses in a close-coiled helical spring is attacked by considering the stresses in an incomplete circular ring. It is found that the cross-section of a ring made from originally circular wire is almost exactly circular, but that other shapes of cross-section would be very considerably modified, especially if the ratio of the coil radius to the wire radius were small. The stresses in a ring of circular cross-section are obtained and good agreement is found with the results of Göhner. Cross-sections of non circular outline are considered with a view to reducing the maximum stress due to a given longitudinal force, but no worthwhile improvement on the circular form is found to be possible.

Introduction

THE problem of the stresses in helical springs is one that has attracted the interest of engineers and mathematicians for many years. Many of the attempts to throw light on the subject have dealt with the problem of a close-coiled spring which, without serious loss of accuracy, can be treated as an incomplete ring or tore, subjected to loading as shown in Fig. 1. The first to make progress towards a solution was J. H. Michell (**1**) in 1900, who solved the problem for rings with oval cross-sections. His method involved finding a function satisfying a differential equation and taking constant values on the boundary of the cross-section. An elementary and less rigorous solution was given by Röver (**2**) in 1913, and in 1931 Göhner (**3**), who appeared to be unaware of Michell's work, produced a solution based on the same differential equation. He, however, went farther than Michell and obtained series solutions of the problem for a variety of cross-sections.

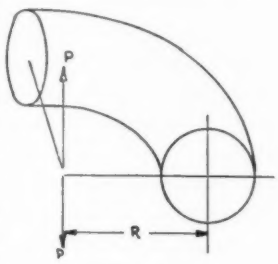

FIG. 1.

In this paper the problem is attacked again, using the same differential equation and boundary conditions as Michell and Göhner, but using a method of solution different from Göhner's, being in a sense an extension of Michell's method. The method is applicable to a wide variety of oval sections.

Consideration is first given to the shape of the cross-sections of springs made from wire originally circular in section. An explanation is given of

the surprising fact that the cross-sections of these springs are almost exactly circular.

The spring of circular section is treated in detail and finally consideration is given to the possibility of reducing the maximum stress by using non-circular sections. It is shown that no worthwhile improvement is to be obtained in this way.

Form of the cross-section

Almost all helical springs in common use are made from wire of which the cross-section was originally circular. In the usual elementary treatment of the problem it is assumed that, in the final form, the cross-section is still circular. This assumption is also made by those who have treated the problem more accurately. In view of the large permanent strains produced in the material during the manufacture of springs, it appeared necessary to examine this point carefully.

Experiment shows that a straight bar of square section when bent to form a circular arc has finally a cross-section very far from square. It was, therefore, with some surprise that the cross-section of a valve spring taken from an aero-engine was found to be almost exactly circular. In this spring the ratio of the radius of the coil to the radius of the wire was about 8. A ground rod of the same section was then bent so that the corresponding ratio was about 3. Again the cross-section was found to be almost circular.

The mathematical solution of the problem of the permanent strain in a bent rod involves considerable difficulty, but an approximate solution may be obtained which gives, in the author's opinion, adequate explanation of what was to him an unexpected result.

When a rod is bent into a circular arc the strain consists of two parts—the elastic or recoverable part and the plastic or permanent part. On releasing the rod the elastic strains disappear almost completely, leaving only the plastic strains. The plastic part of the strain can be assumed to involve no volume change and will correspond to displacements of approximately the *type* given by the equations of elastic bending if Poisson's ratio is taken as 0·5.

In Fig. 2 Ox, Oz are axes in the plane of a cross-section, Ox being in the plane of bending. Then the well-known theory gives the component displacements u, w. Apart from rigid body displacements u and w are given by

$$u = (z^2 - x^2)/4R,$$
$$w = -xz/2R,$$

where R is the radius of curvature of the strained centre line. These formulae are chosen so that the centre of the cross-section is not moved by the straining. It must, of course, be realized that these equations are

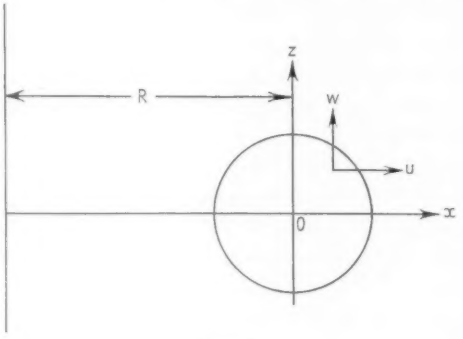

FIG. 2.

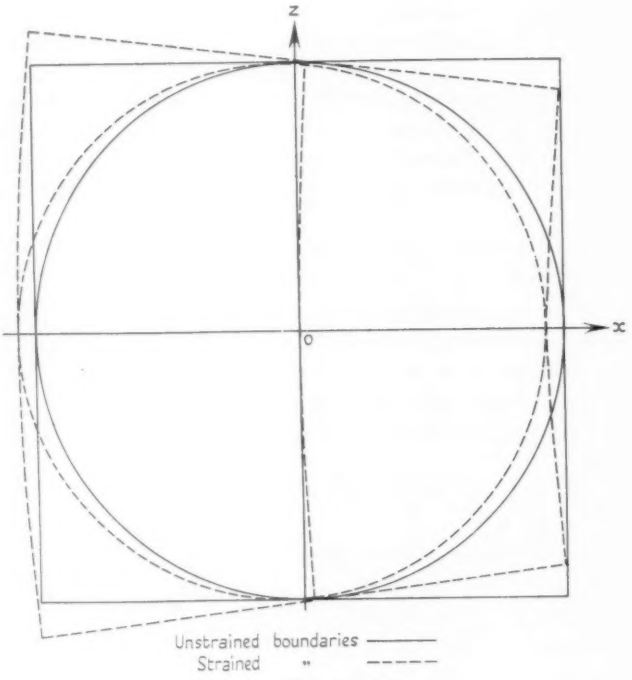

FIG. 3.

only strictly valid if the strains are small. In the problem considered the strains may be by no means small and so the equations can only be expected to give an approximate picture of the form of the distortion.

Fig. 3 shows the original and strained forms of the cross-section using

a ratio $R/a = 4$, a being the radius of the cross-section. In the same diagram are shown the strained and unstrained forms of the cross-section of a square bar deduced in the same way. Experiment gave very fair confirmation of these boundaries. It will be seen that despite the very considerable internal strains the boundary of the cross-section of the bent circular rod is, for all practical purposes, still circular. On this approximate theoretical basis the greatest deviation from the circular form should occur near the diameter at right angles to the plane of bending, but if $R/a = 4$, this only amounts to about 0·5 per cent. of the radius. Measurement of a bent rod indicates that the actual deviation is less than this theoretical value. For larger values of the ratio R/a the deviation will, of course, be smaller still.

It appears that the circular section is the only one which preserves its form in bending. Even elliptical sections will be considerably distorted if the major axis is in the direction Oz and is large compared with the minor axis.

This investigation casts considerable doubt on the usefulness of the investigations into the stresses in springs of square section. To obtain a spring of which the cross-sections are square, the form of the wire used would have to be carefully calculated and would be by no means square.

The twist of a circular ring sector

Using cylindrical polar coordinates (r, θ, z), r denoting the distance from the axis of the ring, z the distance from its middle plane, and θ the angular displacement about the axis of the ring, and following Michell (**1**) and Göhner (**3**), but modifying the notation, it is assumed that the only stresses are $\widehat{\theta r}$ and $\widehat{\theta z}$ and that these stresses are functions of r and z only. Then the only body-stress equation not identically satisfied is

$$\frac{\partial r\widehat{\theta}}{\partial r} + \frac{\partial \widehat{\theta z}}{\partial z} + \frac{2r\widehat{\theta}}{r} = 0, \tag{1}$$

and the compatibility equations give

$$\frac{\partial^2 \widehat{\theta z}}{\partial r^2} + \frac{1}{r}\frac{\partial \widehat{\theta z}}{\partial r} + \frac{\partial^2 \widehat{\theta z}}{\partial z^2} - \frac{\widehat{\theta z}}{r^2} = 0, \tag{2}$$

$$\frac{\partial^2 r\widehat{\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial r\widehat{\theta}}{\partial r} + \frac{\partial^2 r\widehat{\theta}}{\partial z^2} - \frac{4r\widehat{\theta}}{r^2} = 0. \tag{3}$$

Equation (1) is satisfied if

$$r\widehat{\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial z}, \qquad \widehat{\theta z} = -\frac{1}{r^2}\frac{\partial \phi}{\partial r}, \tag{4}$$

where ϕ is a function of r and z, and then equations (2) and (3) lead to

$$\frac{\partial^2\phi}{\partial r^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{3}{r}\frac{\partial\phi}{\partial r}=\text{constant}. \tag{5}$$

The stress $\widehat{\theta n}$ across the boundary is given by

$$\widehat{\theta n}=\frac{\partial z}{\partial s}\widehat{r\theta}-\frac{\partial r}{\partial s}\widehat{z\theta},$$

where s, n denote the tangential and normal directions, and equation (4) then gives

$$\widehat{\theta n}=\frac{1}{r^2}\frac{\partial\phi}{\partial s}.$$

But $\widehat{\theta n}=0$, so that

$$\phi=\text{constant along the boundary}. \tag{6}$$

The problem then consists of finding a solution of equation (5) which satisfies equation (6) on this boundary and which gives finite and continuous stresses within the boundary.

The method of attack used by Göhner involved finding a sequence of functions ϕ_1, ϕ_2,..., each of which is constant on the boundary and such that the function ϕ, defined by

$$\phi=\phi_1+\phi_2+\ldots+\phi_n,$$

approaches a solution of equation (5).

The method used in this paper is in a sense the opposite of this. It consists of finding a sequence of functions such that each is a solution of equation (5) and combining them linearly in such a way that the sum is constant on the boundary. On putting

$$\phi=\psi+A+Bz^2+Cr^2$$

in equation (5), A, B, C being constants and ψ a function of r and z, it is found that

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{3}{r}\frac{\partial\psi}{\partial r}=0.$$

If the cross-section is symmetrical about Ox, then ψ must be even in z and, if it is assumed homogeneous and of even degree $2n$ in r and z, then

$$\psi=\sum_{p=0}^{n}a_{2p}r^{2p}z^{2n-2p}.$$

It follows without difficulty that

$$(2n-2p)(2n-2p-1)a_{2p}=-(2p+2)(2p-2)a_{2p+2}$$

and that $a_0=a_2=0$.

The functions of degrees 4, 6, 8 found in this way are

$$r^4,\quad 6r^4z^2-r^6,\quad 16r^4z^4-16r^6z^2+r^8.$$

It is easily shown that there are no functions of odd degree. Functions of higher even degree could easily be found if required, but for the purpose of dealing with the ring of circular section and with the other sections considered here, only the first two of these functions are necessary.

The complete function ϕ is then taken in the form

$$\phi = A+Br^2+Cz^2+Dr^4+2F(6r^4z^2-r^6) \tag{7}$$

and the boundary is taken to be given by $\phi = 0$.

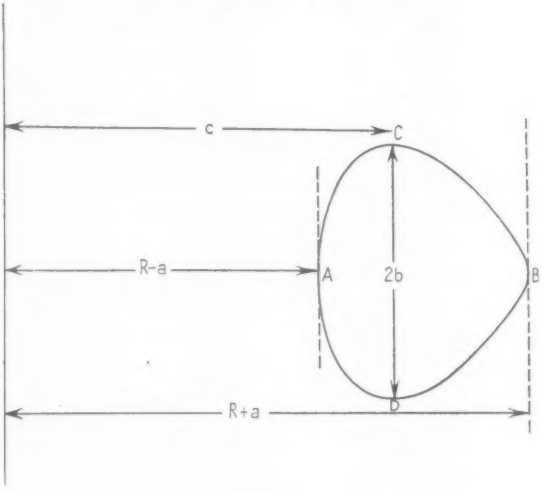

FIG. 4.

If the boundary of the cross-section crosses the axis $z = 0$ at points where $r = R \pm a$, then equation (7) may be rewritten in the form

$$\frac{\phi}{2F} = \{r^2-(R-a)^2\}\{(R+a)^2-r^2\}\{r^2+k\}+Kz^2+6z^2r^4, \tag{8}$$

where K and k are two new constants which may be used to satisfy further conditions. For the purposes of this paper it is most convenient to prescribe the position and magnitude of the greatest width (in the z-direction) of the section (see Fig. 4).

For the circular section this greatest width must occur at $r = R$, $z = \pm a$, so that regarding $\phi = 0$ as the equation of the boundary, the conditions are

$$\text{(i)} \quad \phi = 0, \qquad \text{(ii)} \quad \frac{\partial\phi}{\partial r} = 0, \quad \text{when} \quad r = R \quad \text{and} \quad z = a.$$

The second of these conditions gives

$$2k = a^2-18R^2, \tag{9}$$

and using this result the first condition gives

$$2K = 52R^4 - 20R^2a^2 + a^4. \tag{10}$$

The equation of the boundary now becomes

$$z^2 = \frac{\{r^2-(R-a)^2\}\{(R+a)^2-r^2\}\{18R^2-a^2-2r^2\}}{12r^4+52R^4-20R^2a^2+a^4}. \tag{11}$$

It is now necessary to find out how nearly equation (11) represents a circle. The substitution

$$r = R+x \tag{12}$$

transfers the origin to the centre of the cross-section and the distance ρ of a point on the boundary from the centre is given by

$$\rho^2 = x^2+z^2. \tag{13}$$

On making the substitution (12) in equation (11) and using equation (13), it is found, after some algebraic manipulation, that

$$a^2-\rho^2 = \frac{x^2(a^2-x^2)(80R^2+60Rx+14x^2-a^2)}{64R^4+48R^3x+72R^2x^2-20R^2a^2+48Rx^3+12x^4+a^4}$$

$$= x^2(a^2-x^2)\left[\frac{5}{4R^2}+\frac{6a^2-19x^2}{16R^4}+\ldots\right]. \tag{14}$$

Since $a-\rho$ is small the relation may be written

$$a-\rho = \frac{5x^2(a^2-x^2)}{8aR^2} \tag{15}$$

approximately.

Equation (14) shows that the boundary fits the circle accurately at the four points whose x and z coordinates are $(0, -a)$, $(0, +a)$, $(a, 0)$, $(-a, 0)$.

The greatest error will occur near the points where $x^2 = \frac{1}{2}a^2$, and there $a-\rho = 5a^3/32R^2$ approximately. If $R = 4a$, the error is about 1 per cent. and for larger ratios of R/a becomes rapidly smaller, being about $\frac{1}{4}$ per cent. when $R = 8a$. Thus it is seen that the boundary is circular for all practical purposes.

Using the values of K and k given in equations (9) and (10), equation (8) becomes

$$\frac{\phi}{F} = 12z^2r^4-2r^6+z^2(52R^4-20R^2a^2+a^4)-r^2(38R^4+30R^2a^2)+$$
$$+r^4(22R^2+3a^2)+18R^6-37R^4a^2+20R^2a^4-a^6.$$

Then equations (4) give the stresses

$$\frac{1}{F}\widehat{r\theta} = 24zr^2+\frac{2z}{r^2}(52R^4-20R^2a^2+a^4), \tag{16}$$

$$\frac{1}{F}\widehat{\theta z} = -48z^2r+12r^3-4(22R^2+3a^2)r+\frac{2}{r}(38R^4+30R^2a^2). \tag{17}$$

The resultant of these stresses acting on a cross-section must be equivalent to a force P acting along the axis of the ring. This force P is given by

$$P = \iint \widehat{\theta z}\, dr dz$$

taken over the cross-section.

The boundary of the cross-section is so nearly circular that only a negligible error is introduced by taking the integral throughout the interior of the circle $\rho = a$. This point is referred to again later. The integrals involved are all elementary and the final result is

$$P = \pi F[152R^5 + 44R^3a^2 - 15Ra^4 - (152R^4 + 120a^2R^2)\sqrt{(R^2 - a^2)}].$$

On expanding the square root in powers of a/R this gives

$$\frac{P}{F\pi a^5} = \frac{64R}{a}\left[1 + \frac{49}{128}\left(\frac{a}{R}\right)^2 + \frac{215}{1024}\left(\frac{a}{R}\right)^4 + \text{higher even powers of } (a/R)\right]. \tag{18}$$

An estimate of the error introduced by integrating over the circle instead of over the exact boundary was made by evaluating the integral numerically over the small region between the boundary and the circle in the case $R = 4a$.

The error was found to be less than $\frac{1}{3}$ per cent. and would be still less for larger values of R/a.

The greatest stress occurs at the point $r = R - a$, $z = 0$ and there $\widehat{\theta z} = q$, $\widehat{r\theta} = 0$, where

$$\frac{q}{F} = 128R^2a + 160a^2R + 136a^3 \Big/ \left(1 - \frac{a}{R}\right). \tag{19}$$

From equations (18) and (19) the relation between the transverse force P and the greatest stress q is obtained in the form

$$\frac{q\pi a^2}{P} = \frac{2R}{a} \frac{1 + \frac{5}{4}\frac{a}{R} + \frac{17}{16}\left(\frac{a}{R}\right)^2\left(1 - \frac{a}{R}\right)^{-1}}{1 + \frac{49}{128}\left(\frac{a}{R}\right)^2 + \frac{215}{1024}\left(\frac{a}{R}\right)^4}. \tag{20}$$

Göhner's result, as given by Timoshenko (**4**), is, after an error in sign has been corrected,

$$\frac{q\pi a^2}{P} = \frac{2R}{a} \frac{\left(1 - \frac{a}{R}\right)^{-1} + \frac{1}{4}\frac{a}{R} + \frac{1}{16}\left(\frac{a}{R}\right)^2}{1 + \frac{3}{16}\frac{(a/R)^2}{1 - (a/R)^2}}.$$

The numerical values are set out in Table I.

TABLE I

Table of Max. Stress × Cross-sectional Area/Transverse Force

R/a		4	6	8
$\frac{q\pi a^2}{P}$	Göhner . . .	10·96	14·84	18·75
	Shepherd . .	10·95	14·76	18·69
	Elementary theory .	8	12	16

For values of R/a greater than 8, the formula

$$\frac{q\pi a^2}{P} = \frac{2R}{a} + \frac{5}{2}$$

is of ample accuracy for practical purposes.

The differences between the results obtained by Göhner and those obtained by the author are not significant. They are due to the slightly different method of approximating to the solution and to the deviation from the circular form of the cross-section used by the author. This deviation is of the same type as the actual deviation, but is, as has been found, extremely small.

Applications to sections other than circular

The method used in this paper can be easily applied to the solution of the problem for oval sections other than circular ones. The shape of the oval section may be specified by the dimensions shown in Fig. 4.

Taking the stress function in the form

$$\phi = \{r^2-(R-a)^2\}\{(R+a)^2-r^2\}\{r^2+k\}+Kz^2+6z^2r^4$$

and the boundary to be given by $\phi = 0$, the conditions at A and B are automatically satisfied and it remains to satisfy the conditions at C and D, i.e.

$$\phi = 0, \qquad \frac{\partial\phi}{\partial r} = 0 \quad \text{when} \quad r = c, \qquad z = b.$$

These two conditions give equations to determine k and K. It will be noted that k depends only on c and not on b.

The obvious next step is to find out whether it is possible to reduce the maximum stress corresponding to a given load by changing the shape of the cross-section. The common-sense method of reducing a maximum stress is to increase the amount of material in the highly stressed region. Since the greatest stress occurs at A it appears at first sight likely that a section of the form shown in Fig. 4 with $a = b$ would be more efficient than the circular section of the same overall dimensions. In order to test this supposition, the problem has been solved for the series of dimensions

$$\frac{R}{a} = 4, \quad \frac{b}{a} = 1, \quad \frac{c}{a} = 3{\cdot}6,\ 3{\cdot}7,\ 3{\cdot}8,\ 3{\cdot}9,\ 4{\cdot}0,\ 4{\cdot}1,\ 4{\cdot}2,\ 4{\cdot}3,\ 4{\cdot}4,$$

the total load in each case being obtained by numerical integration. The results are shown graphically in Fig. 5, and it is found that the most efficient section is that in which the greatest width occurs at about $r = 4{\cdot}2a$, i.e. rather nearer to the outside of the ring than to the inside.

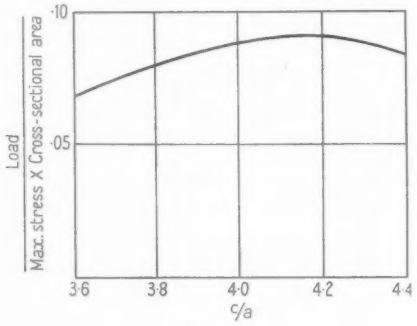

FIG. 5.

This unexpected result may perhaps be explained by the fact that the effective mean radius of the ring is increased or decreased according as the value of c is increased or decreased, and that the effect of this is greater than the effect of the change of the distribution of the material. The greatest change in maximum stress possible is, however, so small that it is not worth while using a non-circular section for this purpose.

The method put forward here could be used without difficulty to deal with the cross-section produced from an originally elliptical rod.

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ISOLATED FORCE PROBLEMS IN TWO-DIMENSIONAL ELASTICITY (II)†

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[Received 6 September 1949]

SUMMARY

In Part I of this paper the complex potential method due to Stevenson (**1**) was applied to solve the problem of one or more isolated forces on the boundary of a hole in an infinite plate. In this part the same method is used to solve the problem of several isolated forces in equilibrium on the boundary of discs of certain shapes.

1. Isolated forces on the boundary of a circular disc with central fixing

If there are a number of forces F_r acting at the points A_r $(z = a_r)$ on the boundary of a circular disc, then they must themselves be in equilibrium

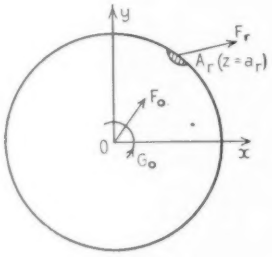

Fig. 1.

or be balanced by external forces applied elsewhere. Suppose this is done by a force and couple nucleus F_0, G_0 at the centre, so that

$$F_0 + \sum F_r = 0, \tag{1.1}$$

$$G_0 + \tfrac{1}{2}i \sum (\bar{F}_r a_r - F_r \bar{a}_r) = 0. \tag{1.2}$$

For the isolated forces on the boundary it is only necessary to assume the complex potentials:

$$\Omega_1 = -\frac{2}{\pi} \sum F_r \log z_r, \qquad z_r = z - a_r,$$

$$\omega_1 = \frac{2z}{\pi} \sum \bar{F}_r \log z_r + \frac{2}{\pi} \sum (F_r \bar{a}_r - \bar{F}_r a_r) \log z_r,$$

† Forms part of a thesis approved for the degree of Ph.D. by the University of London.

and for the force and couple nucleus at the centre,

$$\Omega_2 = \frac{-2F_0 \log z}{\pi(\kappa+1)}, \qquad \omega_2 = \frac{2\kappa\overline{F}_0 z \log z}{\pi(\kappa+1)} + \frac{2iG_0}{\pi}\log z.$$

The complete solution is then given by

$$\Omega = \Omega_0+\Omega_1+\Omega_2, \qquad \omega = \omega_0+\omega_1+\omega_2,$$

where Ω_0 and ω_0 are expansible in positive powers of σ, and Ω and ω satisfy the boundary condition

$$\overline{\Omega}(a^2/z)+(a^2/z)\Omega'(z)+\omega'(z) = k \quad \text{round } z\bar{z} = a^2.$$

Then as $$\log \bar{z}_r = \log(\bar{z}-\bar{a}_r) = \log\left(\frac{-\bar{a}_r z_r}{z}\right)$$

we have

$$\begin{aligned}\overline{\Omega}_1(\bar{z})+\overline{\Omega}_2(\bar{z})+\omega_1'(z)+\omega_2'(z) &= -\frac{2}{\pi}\sum \overline{F}_r\{\log(-\bar{a}_r)+\log z_r-\log z\}- \\ &- \frac{2\overline{F}_0}{\pi(\kappa+1)}(\log a^2-\log z)+\frac{2}{\pi}\sum \overline{F}_r \log z_r+\frac{2}{\pi}\sum \frac{\overline{F}_r z}{z_r}+ \\ &+ \frac{2}{\pi}\sum \frac{(F_r\bar{a}_r-\overline{F}_r a_r)}{z_r}+\frac{2\kappa\overline{F}_0}{\pi(\kappa+1)}(\log z+1)+\frac{2iG_0}{\pi z} \\ &= k'+\frac{2iG_0}{\pi z}+\frac{2}{\pi}\sum \frac{F_r\bar{a}_r}{z_r}.\end{aligned}$$

Also $$\frac{a^2}{z z_r} = \left(\frac{1}{z_r}-\frac{1}{z}\right)\bar{a}_r,$$

so that $$\frac{a^2}{z}\{\Omega_1'(z)+\Omega_2'(z)\} = \frac{2}{\pi}\sum F_r\bar{a}_r\left(\frac{1}{z}-\frac{1}{z_r}\right)-\frac{2a^2F_0}{\pi(\kappa+1)z^2}.$$

These equations reduce the boundary condition to

$$\overline{\Omega}_0\left(\frac{a^2}{z}\right)+\frac{a^2}{z}\Omega_0'(z)+\omega_0'(z)+\frac{1}{\pi z}\sum(\overline{F}_r a_r+F_r\bar{a}_r)-\frac{2a^2F_0}{\pi(\kappa+1)z^2} = k'',$$

which can be satisfied by a suitable choice of k'', together with $\omega_0 = 0$ and $\Omega = Az+Bz^2$, where A is real; for the boundary condition becomes

$$\frac{Aa^2}{z}+\frac{\overline{B}a^4}{z^2}+\frac{a^2}{z}(A+2Bz)+\frac{1}{\pi z}\sum(\overline{F}_r a_r+F_r\bar{a}_r)-\frac{2a^2F_0}{\pi(\kappa+1)z^2} = k'',$$

whence $$A = -\frac{1}{2\pi a^2}\sum(\overline{F}_r a_r+F_r\bar{a}_r), \qquad B = \frac{2\overline{F}_0}{\pi a^2(\kappa+1)},$$

so that the complete solution is

$$\Omega = -\frac{2}{\pi}\sum F_r\log z_r-\frac{2F_0\log z}{\pi(\kappa+1)}-\frac{1}{2\pi a^2 z}\sum(\overline{F}_r a_r+F_r\bar{a}_r)+\frac{2\overline{F}_0 z^2}{\pi a^2(\kappa+1)}, \quad (1.3)$$

$$\omega = \frac{2z}{\pi}\sum\overline{F}_r\log z_r+\frac{2}{\pi}\sum(F_r\bar{a}_r-\overline{F}_r a_r)\log z_r+\frac{2\kappa\overline{F}_0 z\log z}{\pi(\kappa+1)}+\frac{2iG_0}{\pi}\log z. \quad (1.4)$$

The peripheral stress

As before $2(\breve{nn}+\breve{ss}) = \Omega'(z)+\overline{\Omega}'(\bar{z})$, and $\breve{nn} = 0$ round the boundary, so that the peripheral stress $\breve{ss}$ is given by

$$2\breve{ss} = \frac{2(F_0\bar{z}+\overline{F}_0 z)}{\pi a^2(\kappa+1)} - \frac{1}{\pi a^2}\sum(\overline{F}_r a_r + F_r\bar{a}_r) - \frac{2}{\pi}\sum\frac{F_r\bar{z}_r+\overline{F}_r z_r}{2a^2-a_r\bar{z}-\bar{a}_r z},$$

and as

$$\begin{aligned} F_r\bar{z}_r+\overline{F}_r z_r &= F_r(\bar{z}-\bar{a}_r)+\overline{F}_r(z-a_r) \\ &= F_r\bar{z}+\overline{F}_r z-(F_r\bar{a}_r+\overline{F}_r a_r) \end{aligned}$$

then

$$\breve{ss} = \frac{F_0\bar{z}+\overline{F}_0 z}{\pi a^2(\kappa+1)} - \frac{1}{2a^2\pi}\sum(\overline{F}_r a_r+F_r\bar{a}_r) - \frac{1}{\pi}\sum\frac{F_r\bar{z}+\overline{F}_r z-(F_r\bar{a}_r+\overline{F}_r a_r)}{2a^2-a_r\bar{z}-\bar{a}_r z}.$$

Diametrically opposite forces

If there are two forces $F_1 = -iR$ at $z = ia$, and $F_2 = iR$ at $z = -ia$, F_0 and G_0 being zero, then

$$\overline{F}_1 a_1+F_1\bar{a}_1 = -2Ra, \qquad \overline{F}_2 a_2+F_2\bar{a}_2 = -2Ra$$

and therefore

$$\sum(\overline{F}_r a_r+F_r\bar{a}_r)/(2\pi a^2) = -2R/(\pi a).$$

Also

$$\{F_1\bar{z}+\overline{F}_1 z-(F_1\bar{a}_1+\overline{F}_1 a_1)\}/(2a^2-a_1\bar{z}-\bar{a}_1 z) = R/a,$$

$$\{F_2\bar{z}+\overline{F}_2 z-(F_2\bar{a}_2+\overline{F}_2 a_2)\}/(2a^2-a_2\bar{z}-\bar{a}_2 z) = R/a,$$

therefore

$$\breve{ss} = 0+\frac{2R}{\pi a}-\frac{2R}{\pi a} = 0$$

checking with the well-known result.

2. Curvilinear polygonal disc under isolated boundary forces

The transformation $z = c\sigma(1+\lambda\sigma^{+n})$, where n is integral and greater than zero, and c, λ are real positive constants, maps the interior of a curvilinear polygon on the interior of a unit circle in a complex σ-plane, and is everywhere conformal if $z'(\sigma) \neq 0$ within the unit circle.

Also as $z'(\sigma) = c[1+(n+1)\lambda\sigma^n]$ and $z'(\sigma) = 0$ gives $\sigma^n = -\dfrac{1}{\lambda(n+1)}$, then for $0 \leqslant (n+1)\lambda \leqslant 1$ there will be no critical points inside the unit circle.

This transformation is of particular interest in two cases (i) when $n = 2$, the curvilinear polygon becoming a 'dumb bell' shape somewhat resembling that of the Standard British and French Cement Briquettes, and (ii) when n is large and the disc also has central fixing, as we now have an approximation to the problem of a cog-wheel (with shallow teeth).

Case (i) In this case $z = c\sigma(1+\lambda\sigma^2)$ and $\lambda \leqslant \frac{1}{3}$. Varying λ from $\frac{1}{3}$ to zero results in a succession of shapes ranging from a two-cusped epicycloid

to a circle (Fig. 2), the ratio of the major and minor axes varying between 2 and 1 in the extreme cases. (The cement briquette, however, though much of the same shape, has a ratio of 3:1.)

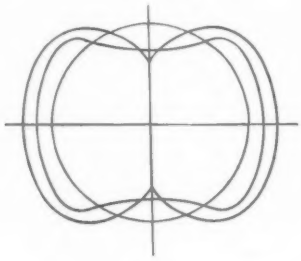

FIG. 2.

If there are a number of complex forces $F_r = Re^{i\beta_r}$ at the points $z = z_r$ corresponding to $\sigma = \sigma_r = e^{i\alpha_r}$, then from the equilibrium of the external forces on the disc

$$\sum F_r = 0, \qquad \sum \overline{F}_r z_r = \sum F_r \bar{z}_r \tag{2.1}$$

and the complex potentials giving these isolated forces are

$$\Omega_1 = -\frac{2}{\pi}\sum F_r \log \zeta_r \quad \text{where} \quad \zeta_r = \sigma - \sigma_r,$$

$$\omega_1 = \frac{2z}{\pi}\sum \overline{F}_r \log \zeta_r + \frac{2}{\pi}\sum (F_r \bar{z}_r - \overline{F}_r z_r)\log \zeta_r.$$

The complex potentials for the problem are then of the form

$$\Omega = \Omega_0 + \Omega_1, \qquad \omega = \omega_0 + \omega_1,$$

where $\Omega_0(\sigma)$ and $\omega_0(\sigma)$ are expansible in positive powers of σ and are chosen to satisfy the boundary condition

$$z'(\sigma)\overline{\Omega}_0(\sigma^{-1}) + \bar{z}(\sigma^{-1})\Omega'(\sigma) + \omega_0'(\sigma) + f(\sigma) = 0,$$

where

$$f(\sigma) = z'(\sigma)\left\{-\frac{2}{\pi}\sum \overline{F}_r \log\frac{\bar{\zeta}_r}{\zeta_r} - k\right\} - \frac{2}{\pi}\bar{z}(\sigma^{-1})\sum \frac{F_r}{\zeta_r} + \\ + \frac{2}{\pi}\sum \frac{\overline{F}_r z}{\zeta_r} + \frac{2}{\pi}\sum \frac{F_r \bar{z}_r - \overline{F}_r z_r}{\zeta_r}. \tag{2.2}$$

As $\bar{\zeta}_r/\zeta_r = -1/(\sigma_r\sigma)$, then $\sum \overline{F}_r \log \bar{\zeta}_r/\zeta_r$ is a constant; also

$$(z - z_r)/\zeta_r = c(1 + \lambda\sigma_r^2 + \lambda\sigma_r\sigma + \lambda\sigma^2)$$

and

$$(\bar{z} - \bar{z}_r)/\zeta_r = -\bar{z}_r\sigma^{-1} - \lambda c\sigma^{-2}\sigma_r^{-2} - \lambda c\sigma^{-3}\sigma_r^{-3},$$

whence (2.2) gives, after some reduction,

$$f(\sigma) = (2c/\pi)\sum \overline{F}_r(\lambda\sigma_r^2 + \lambda\sigma_r\sigma) + (2c/\pi)\sum F_r(\lambda\sigma^{-2}\sigma_r^{-2} + \lambda\sigma^{-3}\sigma_r^{-3}) + \\ + (2/\pi\sigma)\sum F_r\bar{z}_r,$$

so that $f(\sigma)$ contains powers of σ ranging from σ^{+1} to σ^{-3} only and we accordingly take

$$\Omega_0 = \sum_1^3 A_r \sigma^r, \qquad \omega_0 = c \sum_1^2 B_r \sigma^r,$$

then

$$\begin{aligned} z'(\sigma)\overline{\Omega}_0(\sigma^{-1}) + \bar{z}(\sigma^{-1})\Omega_0'(\sigma) + \omega_0'(\sigma) \\ = c\{[(A_1+\bar{A}_1)+3\lambda(A_3+\bar{A}_3)]\sigma^{-1} + (\bar{A}_2+2\lambda A_2)\sigma^{-2} + \\ + (\bar{A}_3+\lambda A_1)\sigma^{-3} + (3\lambda\bar{A}_2+2A_2+B_1) + (3\lambda\bar{A}_1+3A_3+2B_2)\sigma\}; \quad (2.3) \end{aligned}$$

as this must be equal to $-f(\sigma)$ it gives five equations for the five constants A_1, A_2, A_3, B_1, B_2, namely,

$$\begin{aligned} (A_1+\bar{A}_1)+3\lambda(A_3+\bar{A}_3) &= (-2/\pi c)\sum F_r \bar{z}_r, \\ \bar{A}_2+2\lambda A_2 &= (-2\lambda/\pi)\sum F_r \sigma_r^{-2}, \\ \bar{A}_3+\lambda A_1 &= (-2\lambda/\pi)\sum F_r \sigma_r^{-3}, \\ 3\lambda\bar{A}_2+2A_2+B_1 &= (-2\lambda/\pi)\sum \overline{F}_r \sigma_r^2, \\ 3\lambda\bar{A}_1+3A_3+2B_2 &= (-2\lambda/\pi)\sum \overline{F}_r \sigma_r, \end{aligned}$$

whence solving

$$A_1 = \frac{3\lambda^2}{\pi(1-3\lambda^2)}\sum (F_r\sigma_r^{-3}+\overline{F}_r\sigma^3) - \frac{\sum F_r\bar{z}_r}{\pi c(1-3\lambda^2)},$$

$$A_2 = \frac{2\lambda}{\pi(1-4\lambda^2)}\sum (2\lambda F_r\sigma_r^{-2} - \overline{F}_r\sigma_r^2),$$

$$A_3 = -\frac{2\lambda}{\pi}\sum \overline{F}_r\sigma_r^3 - \frac{1}{2\pi(1-3\lambda^2)}[\lambda^2(6\lambda-1)\sum (F_r\sigma_r^{-3}+\overline{F}_r\sigma_r^3) - \\ -\lambda\sum (F_r\sigma_r^{-1}+\overline{F}_r\sigma_r)],$$

$$B_1 = \frac{-2\lambda}{\pi(1-4\lambda^2)}[\sum \lambda F_r\sigma_r^{-2} - (1+10\lambda^2)\sum \overline{F}_r\sigma_r^2],$$

$$B_2 = -\frac{\lambda}{\pi}\sum \overline{F}_r(\sigma_r - 3\sigma_r^3).$$

Disc under tension

If there are two equal, opposite, and horizontal forces

$$F_1 = T \text{ at } \sigma_1 = 1, \qquad F_2 = -T \text{ at } \sigma_2 = -1,$$

then substituting in the above equations

$$\Omega_0 = A_1\sigma + A_3\sigma^3, \qquad \omega_0 = B_2\sigma^2, \qquad \Omega_1 = -\frac{2T}{\pi}\log\left(\frac{\sigma-1}{\sigma+1}\right),$$

where $$A_1 = \frac{2T(6\lambda^2-\lambda-1)}{\pi(1-3\lambda^2)} \quad \text{and} \quad A_3 = \frac{2T\lambda(\lambda-1)}{1-3\lambda^2},$$

so that the peripheral stress $\breve{ss}$, given by

$$\breve{ss} = \text{real part of } \Omega'(\sigma)/z'(\sigma),$$

reduces to

$$\breve{ss} = \frac{4T}{\pi c}\,\frac{\lambda(1-3\lambda)}{(1-3\lambda^2)(1+9\lambda^2+6\lambda\cos 2\eta)}$$

(see Fig. 5). This vanishes all round the disc for the two cases $\lambda = \frac{1}{3}$ and $\lambda = 0$, i.e. the two-cusped epicycloid and the circle (which agrees with the known result).

Also at a point $\sigma = ia$ (a point on the mid-section of the disc) it is readily shown that

$$\begin{aligned}\breve{x}\breve{x}+\breve{y}\breve{y} &= \Omega'(\sigma)/z'(\sigma)+\overline{\Omega}'(\bar{\sigma})/\bar{z}'(\bar{\sigma})\\ &= [A_1+3A_3\sigma^2-4T/\pi(\sigma^2-1)]/[c(1+3\lambda\sigma^2)]\\ &= 2[-2T/\pi+2Ta^2+4T/\pi(1+a^2)]/c(1-a^2)\end{aligned}$$

(see Fig. 6). This is infinite at $a = \pm 1$ (the ends of the minor axis) and has the value $4T/\pi c$ at $a = 0$ (the centre).

Disc under briquette conditions

If there are four symmetrically disposed isolated forces, as in Fig. 3,

$$\begin{aligned}F_1 &= Re^{-i\beta} \quad \text{at} \quad \sigma = \sigma_1 = e^{i\alpha},\\ F_2 &= -\overline{F}_1 \quad \text{at} \quad \sigma = \sigma_2 = -\sigma^{-1},\\ F_3 &= -F_1 \quad \text{at} \quad \sigma = -\sigma_1,\end{aligned}$$

and

$$F_4 = \overline{F}_1 \quad \text{at} \quad \sigma = \bar{\sigma}_1 = \sigma_1^{-1}.$$

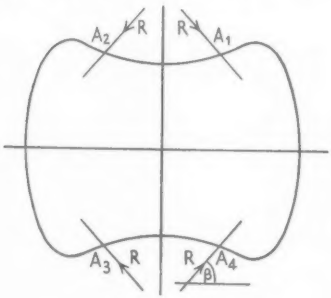

FIG. 3.

Then substituting in the above relations and simplifying, the peripheral stress is found to be given by

$$\breve{ss} = \breve{ss}_0+\breve{ss}_1,$$

where

$$\begin{aligned}\breve{ss}_0 = -4R[&[(1-9\lambda^2)\{\cos(\alpha-\beta)+\lambda\cos(3\alpha-\beta)\}+12\lambda^2\cos(3\alpha+\beta)]+\\ &+\cos 2\eta[3\lambda(1-3\lambda)\cos(\alpha-\beta)+3\lambda^2(1-3\lambda)\cos(3\alpha-\beta)+\\ &+18\lambda^2(1-\lambda)\cos(3\alpha+\beta)]]/[\pi c(1-3\lambda^2)(1+9\lambda^2+6\lambda\cos 2\eta)]\end{aligned}$$

and

$$\breve{ss}_1 = \frac{2R[\sin(\beta+\eta)\sin(\eta+\alpha)+\sin(\eta-\beta)\sin(\eta-\alpha)]}{\pi c(1+9\lambda^2+6\lambda\cos 2\eta)\sin(\eta-\alpha)\sin(\eta+\alpha)} +$$

$$+\frac{6\lambda R[\sin(3\eta+\alpha)\sin(\eta-\beta)\sin^2(\eta+\alpha)+\sin(\eta+\beta)\sin(3\eta-\alpha)\sin^2(\eta-\alpha)]}{\pi c(1+9\lambda^2+6\lambda\cos 2\eta)\sin^2(\eta-\alpha)\sin^2(\eta+\alpha)},$$

which in the particular case $\lambda = \frac{1}{3}$, $\alpha = \beta = \frac{1}{3}\pi$ further reduces to $\breve{ss} = R/\pi c$, i.e. the peripheral stress is constant round the boundary.

Also $\breve{x}\breve{x}+\breve{y}\breve{y} = \Omega'(\sigma)/z'(\sigma)+\overline{\Omega}'(\bar{\sigma})/\bar{z}'(\bar{\sigma})$, and at $\sigma = ia$ (a point on the mid-section) this reduces to (see Fig. 7)

$$\breve{x}\breve{x}+\breve{y}\breve{y} = R[-18+4(2a^2-1)/(a^4-a^2+1)]/[\pi c(1-a^2)].$$

Case (ii) In this case $z = c\sigma(1+\lambda\sigma^{+n})$ and $\lambda \leqslant 1/(n+1)$, and we suppose there to be isolated boundary forces $F_r = R_r e^{i\beta_r}$ at the points $z = z_r$ (corresponding to $\sigma = \sigma_r = e^{i\alpha_r}$ in the σ-plane), balanced by a force and couple nucleus at the origin given by F_0 and G_0, so that

$$F_0+\sum F_r = 0, \qquad G_0+\tfrac{1}{2}i\sum(\overline{F}_r z_r - F_r\bar{z}_r) = 0.$$

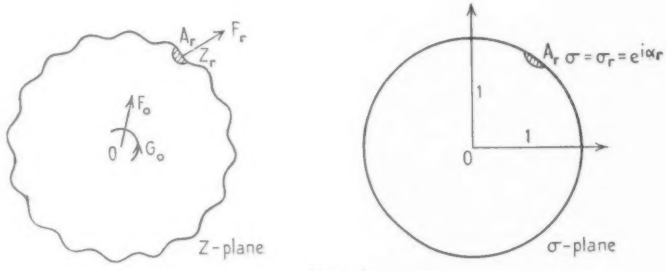

FIG. 4.

The appropriate isolated forces F_r at z_r are given by the complex potentials

$$\Omega_1 = -\frac{2}{\pi}\sum F_r \log\zeta_r \quad \text{where} \quad \zeta_r = \sigma-\sigma_r \tag{2.4}$$

$$\omega_1 = \frac{2z}{\pi}\sum \overline{F}_r\log\zeta_r+\frac{2}{\pi}\sum(F_r\bar{z}_r-\overline{F}_r z_r)\log\zeta_r, \tag{2.5}$$

where $$z_r = c(\sigma_r+\lambda\sigma_r^{n+1}) = \text{constant}.$$

Also the force and couple at the origin O are satisfied by the complex potentials

$$\Omega_2 = -\frac{2F_0\log\sigma}{\pi(\kappa+1)}, \tag{2.6}$$

$$\omega_2 = \frac{2\kappa\overline{F}_0 z\log\sigma}{\pi(\kappa+1)}-\frac{1}{\pi}\sum(F_r\bar{z}_r-\overline{F}_r z_r)\log\sigma. \tag{2.7}$$

The problem is therefore satisfied by the complex potentials

$$\Omega = \Omega_0+\Omega_1+\Omega_2, \qquad \omega = \omega_0+\omega_1+\omega_2,$$

where Ω_0 and ω_0 are to be expressed in positive powers of σ only, and must be chosen to satisfy the boundary condition:

$$z'(\sigma)\{\overline{\Omega}(\sigma^{-1})-k\}+\bar{z}(\sigma^{-1})\Omega'(\sigma)+\omega'(\sigma) = 0.$$

Now,

$$\overline{\Omega}_1(\sigma^{-1})+\overline{\Omega}_2(\sigma^{-1})-k = -\frac{2}{\pi}\sum \overline{F}_r \log \bar{\zeta}_r - k + \frac{2\overline{F}_0 \log \sigma}{\pi(\kappa+1)},$$

and,

$$\log \bar{\zeta}_r = \log\left(-\frac{\bar{\sigma}_r \zeta_r}{\sigma}\right).$$

Also,

$$\omega_1'(\sigma)+\omega_2'(\sigma) = \frac{2\kappa\overline{F}_0}{\pi(\kappa+1)}\left\{z'(\sigma)\log\sigma+\frac{z}{\sigma}\right\}-\frac{1}{\pi\sigma}\sum (F_r\bar{z}_r-\overline{F}_r z_r)+$$

$$+\frac{2z'(\sigma)}{\pi}\sum \overline{F}_r\log\zeta_r+\frac{2z}{\pi}\sum\frac{\overline{F}_r}{\zeta_r}+\frac{2}{\pi}\sum\frac{F_r\bar{z}_r-\overline{F}_r z_r}{\zeta_r},$$

so that the boundary condition becomes

$$z'(\sigma)\overline{\Omega}_0(\sigma^{-1})+\bar{z}(\sigma^{-1})\Omega_0'(\sigma)+\omega_0'(\sigma)+f(\sigma) = 0,$$

where

$$f(\sigma) = z'(\sigma)\left\{-\frac{2}{\pi}\sum\overline{F}_r\log\frac{\zeta_r}{\sigma}-k'+\frac{2}{\pi}\sum\overline{F}_r\log\zeta_r+\frac{2\kappa F_0\log\sigma}{\pi(\kappa+1)}+\right.$$

$$\left.+\frac{2\overline{F}_0\log\sigma}{\pi(\kappa+1)}\right\}+\bar{z}(\sigma^{-1})\left\{-\frac{2}{\pi}\sum\frac{F_r}{\zeta_r}-\frac{2F_0}{\pi(\kappa+1)\sigma}\right\}-\frac{2\kappa\overline{F}_0 z}{\pi(\kappa+1)\sigma}-$$

$$-\frac{1}{\pi\sigma}\sum(F_r\bar{z}_r-\overline{F}_r z_r)+\frac{2z}{\pi}\sum\frac{\overline{F}_r}{\zeta_r}+\frac{2}{\pi}\sum\frac{F_r\bar{z}_r-\overline{F}_r z_r}{\zeta_r}.$$

The logarithmic terms cancel and the term in k' can be neglected as it could always be eliminated by adding the rigid body displacement corresponding to $\omega = k'z$, so that $f(\sigma)$ reduces to

$$f(\sigma) = \frac{2}{\pi}\sum\{\overline{F}_r(z-z_r)-F_r(\bar{z}-\bar{z}_r)\}/\zeta_r-\frac{1}{\pi\sigma}\sum(F_r\bar{z}_r-\overline{F}_r z_r)+\frac{2(\overline{F}_0\kappa z-F_0\bar{z})}{\pi(\kappa+1)\sigma}.$$

Now as $z/\sigma = c(1+\lambda\sigma^n)$ and $\bar{z}/\sigma = c(\sigma^{-2}+\lambda\sigma^{-n-2})$ then

$$z-z_r = c(\sigma-\sigma_r)\{1+\lambda(\sigma^{n+1}-\sigma_r^{n+1})/(\sigma-\sigma_r)\}$$

so that $$(z-z_r)/\zeta_r = z/\sigma+\lambda c(\sigma_r\sigma^{n-1}+\sigma_r^2\sigma^{n-2}+\ldots+\sigma_r^n);$$

also $$(\bar{z}-\bar{z}_r)/\zeta_r = -\bar{z}_r/\sigma-\lambda c(\sigma_r^{-1}\sigma^{-n+1}+\sigma_r^{-2}\sigma^{-n}+\ldots+\sigma_r^{-n}\sigma^{-2})$$

and so $f(\sigma)$ can be written

$$f(\sigma) = \frac{2\lambda c}{\pi}\sum \overline{F}_r(\sigma_r\sigma^{n-1}+\sigma_r^2\sigma^{n-2}+\ldots+\sigma_r^n)+$$

$$+\frac{2\lambda c}{\pi}\sum F_r(\sigma_r^{-1}\sigma^{-n-1}+\sigma_r^{-2}\sigma^{-n}+\ldots+\sigma_r^{-n}\sigma^{-2})+$$

$$+\frac{1}{\pi\sigma}\sum (F_r\bar{z}_r+\overline{F}_r z_r)-\frac{2\kappa}{\pi(\kappa+1)}[\overline{F}_0 c(1+\lambda\sigma^n)+F_0 c(\sigma^{-2}+\lambda\sigma^{-n-2})], \quad (2.8)$$

in which the powers of σ range from n to $-(n+2)$. We therefore take

$$\Omega_0 = \sum_1^{n+2} A_r\sigma^{+r}, \qquad \omega_0 = c\sum_0^n B_r\sigma^{r+1},$$

and the boundary condition becomes

$$c[1+\lambda(n+1)\sigma^n]\sum_1^{n+2}\bar{A}_r\sigma^{-r}+c[\sigma^{-1}+\lambda\sigma^{-n-1}]\sum_1^{n+2} rA_r\sigma^{r-1}+$$

$$+c\sum_1^n (r+1)B_r\sigma^r+f(\sigma) = 0.$$

This can be written

$$f(\sigma)/c+\bar{A}_{n+2}\sigma^{-(n+2)}+\{\lambda(n+2)A_{n+2}+B_0+2A_2+\lambda(n+1)\bar{A}_n\}+$$

$$+\sigma^{-1}\{\bar{A}_1+\lambda(n+1)(A_{n+1}+\bar{A}_{n+1})\}+\sigma^{-2}\{\lambda(n+1)\bar{A}_{n+2}+\lambda nA_n+\bar{A}_2\}+$$

$$+\sigma^n\{(n+1)B_n+(n+2)A_{n+2}\}+\sum_3^{n+1}\{\bar{A}_r+\lambda(n+2-r)A_{n+2-r}\}\sigma^{-r}+$$

$$+\sum_1^{n-1}\{(r+1)B_r+(r+2)A_{r+2}+\lambda(n+1)\bar{A}_{n-r}\}\sigma^r = 0. \quad (2.9)$$

Comparing equations (2.8) and (2.9) we then have the following equations for the various coefficients

$$\bar{A}_{n+2} = \frac{2\kappa\lambda F_0}{\pi(\kappa+1)},$$

$$B_0+2A_2+\lambda(n+1)\bar{A}_n+\lambda(n+2)A_{n+2} = \frac{2\kappa\overline{F}_0}{\pi(\kappa+1)}-\frac{2\lambda}{\pi}\sum \overline{F}_s\sigma_s^n,$$

$$\bar{A}_1+\lambda(n+1)(A_{n+1}+\bar{A}_{n+1}) = -\frac{1}{\pi c}\sum (F_s\bar{z}_s+\overline{F}_s z_s),$$

$$\lambda(n+1)\bar{A}_{n+2}+\lambda nA_n+\bar{A}_2 = \frac{2\kappa F_0}{\pi(\kappa+1)}-\frac{2\lambda}{\pi}\sum F_s\sigma_s^{-n},$$

$$(n+1)B_n+(n+2)A_{n+2} = \frac{2\kappa\lambda\overline{F}_0}{\pi(\kappa+1)},$$

$$\bar{A}_r+\lambda(n+2-r)A_{n+2-r} = -\frac{2\lambda}{\pi}\sum \overline{F}_s\sigma_s^{-n-2+r} \quad (3\leqslant r\leqslant n+1),$$

$$(r+1)B_r+(r+2)A_{r+2}+\lambda(n+1)\bar{A}_{n-r} = -\frac{2\lambda}{\pi}\sum F_s\sigma_s^{n-r} \quad (1\leqslant r\leqslant n-1),$$

and finally solving these equations, the A-coefficients are found to be

$$A_1[1-2\lambda^2(n+1)] = \frac{2\lambda^2(n+1)}{\pi}\sum (F_s\sigma_s^{-1}+\overline{F}_s\sigma_s)-\frac{1}{\pi c}\sum (F_s\bar{z}_s+\overline{F}_s z_s),$$

$$A_2(1-2n\lambda^2) = \frac{2\kappa\overline{F}_0}{\pi(\kappa+1)}[1-\lambda^2(n+1)]-\frac{2\lambda}{\pi}\sum F_s\sigma^{-n}+\frac{2\lambda^2 n}{\pi}\sum F_s\sigma_s^{-2},$$

$$\lambda n A_n = \frac{2\kappa F_0}{\pi(\kappa+1)}[1-\lambda^2(n+1)]-\frac{2\lambda}{\pi}\sum F_s\sigma^{-n}-\frac{2\kappa F_0[1-\lambda^2(n+1)]}{\pi(\kappa+1)(1-2n\lambda^2)}+$$
$$+\frac{1}{1-2n\lambda^2}\left[\frac{2\lambda^2 n}{\pi}\sum \overline{F}_s\sigma_s^2-\frac{2\lambda}{\pi}\sum \overline{F}_s\sigma_s^n\right],$$

$$A_{n+1} = -\frac{2\lambda}{\pi}\sum \overline{F}_s\sigma_s+\frac{\lambda}{\pi c}\sum (F_s\bar{z}_s+\overline{F}_s z_s)-\frac{2\lambda^3(n+1)}{\pi}\sum (F_s\sigma_s^{-1}+\overline{F}_s\sigma_s),$$

$$A_{n+2} = \frac{2\kappa\lambda\overline{F}_0}{\pi(\kappa+1)},$$

$$A_r = \frac{2\lambda}{\pi[1-\lambda^2(n+2-r)r]}\sum \{\lambda(n+2-r)F_s-\overline{F}_s\sigma_s^{n+2}\}\sigma_s^{-r}$$
$$(r=3,4,\ldots,n-1),$$

and substitution in the above equations will then give the B-coefficients if required.

The cog-wheel

If there is a single force $+iF$ at the point $\sigma_1 = 1$, and a force $-iF$ and couple G at the origin, we have the case of a cog-wheel with very shallow teeth under the action of a force and couple at the axle and a force at the actuated cog.

In this case,

$$\Omega_1 = -\frac{2iF}{\pi}\log(\sigma-1), \qquad \Omega_2 = \frac{2iF}{\pi(\kappa+1)}\log\sigma,$$

$$\omega_1 = -\frac{2iFz}{\pi}\log(\sigma-1)+\frac{4iFc(1+\lambda)}{\pi}\log(\sigma-1),$$

$$\omega_2 = \frac{2\kappa iFz}{\pi(\kappa+1)}\log\sigma-\frac{2iFc(1+\lambda)}{\pi}\log(\sigma-1).$$

Also substituting in the various equations for the constants:

$$A_1 = 0, \qquad A_2 = \frac{2iF[\kappa(1-\lambda)-\lambda(1-\lambda n)]}{\pi(\kappa+1)(1-2n\lambda^2)}, \qquad A_{n+1} = \frac{2\lambda iF}{\pi},$$

$$A_{n+2} = \frac{2\kappa\lambda iF}{\pi(\kappa+1)}, \qquad A_r = \frac{2\lambda iF[\lambda(n+2-r)+1]}{\pi[1-\lambda^2 r(n+2-r)]},$$

$$\lambda n A_n = -\frac{2\kappa iF[1-\lambda^2(n+1)]}{\pi(\kappa+1)}-\frac{2\lambda iF}{\pi}+\frac{2iF[\lambda(1-\lambda n)(\kappa+1)+\kappa(1-\lambda^2 n-\lambda^2)]}{\pi(\kappa+1)(1-2n\lambda^2)},$$

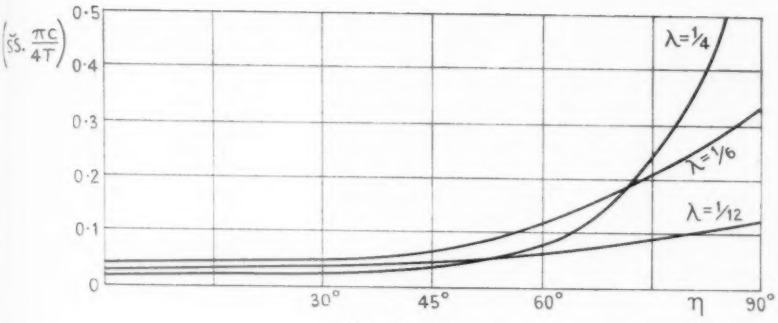

FIG. 5.

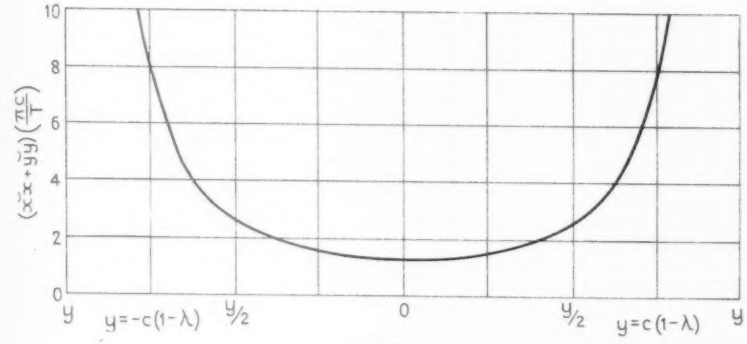

FIG. 6.

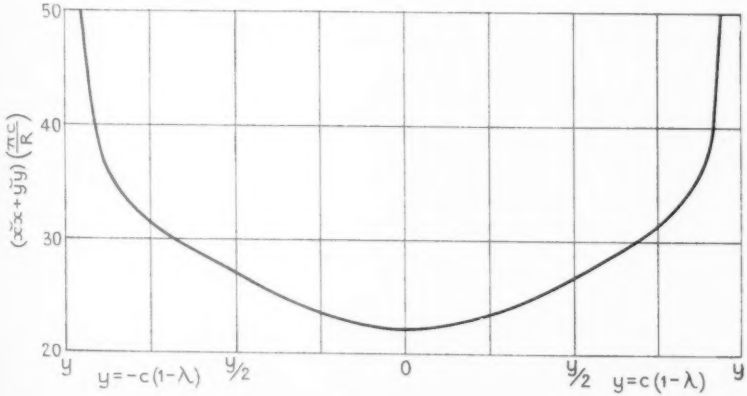

FIG. 7.

so that

$$\Omega_0 = A_1\sigma + A_2\sigma^2 + \sum_3^{n-1} A_r\sigma^r + A_n\sigma^n + A_{n+1}\sigma^{n+1} + A_{n+2}\sigma^{n+2},$$

and the peripheral stress is given by

$$\widetilde{ss} = \text{real part of: } \left[\frac{-2iF}{\pi(\sigma-1)} + \frac{2iF}{\pi(\kappa+1)\sigma} + 2A_2\sigma + nA_n\sigma^{n-1} + \right.$$
$$+ \frac{2\lambda iF}{\pi}(n+1)\sigma^n + \frac{2\kappa\lambda iF}{\pi(\kappa+1)}(n+2)\sigma^{n+1} +$$
$$\left. + \frac{r\sigma^{n-1}(2\lambda iF/\pi)}{c[1+(n+1)\lambda\sigma^n]} \sum_3^{n-1} \frac{1+\lambda(n+2-r)}{1-\lambda^2 r(n+2-r)}\right].$$

I wish to express my further thanks to Mr. A. C. Stevenson for many helpful suggestions in connexion with the above problems.

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1. A. C. Stevenson, 'Complex potentials in 2-dimensional elasticity', *Proc. Roy. Soc.* A, **184** (1945), 129.

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